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Minimum principle of complementary energy of cable networks by using second-order cone programming

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Abstract

The minimum principle of complementary energy is established for cable networks involving only stress components as variables in geometrically nonlinear elasticity. It is rather amazing that the complementary energy always attains minimum value at the equilibrium state irrespective of the stability of cable networks, contrary to the fact that only the stationary principles have been presented for elastic trusses and continua even in the case of stable equilibrium state. In order to show the strong duality between the minimization problems of total potential energy and complementary energy, the convex formulations of these problems are investigated, which can be embedded into a primal–dual pair of second-order cone programming problems. The existence and uniqueness of solution are also investigated for the minimization problem of complementary energy.

Keywords: Nonlinear elasticity; Cable network; Complementary energy; Second-order cone programming; Duality theory

1. Introduction

Among the various variational principles in static mechanics, the complementary energy principle in large deformation elasticity has raised several interesting discussions since the first contribution by Hellinger (1914). Under assumption of small deformation, it is well known that the complementary energy principle contains only stress components as independent variables, whereas the displacements are independent variables in the total potential energy principle. This beautiful symmetric property, however, seems to break if we allow large rotation, because the well-known Hellinger–Reissner principle involves the unknown displacement as well as the second Piola–Kirchhoff stress. Coupling of the unknown stress and displacement prevents us from development of force method allowing finite rotation. From both theoretical and practical points of view, it has been a challenging task to formulate the complementary energy principle only in terms of stress components.

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Koiter (1976) made the important contribution through the excellent survey of related works until 70s. The complementary energy formulation using the first Piola–Kirchhoff stress along with the deformation gradient was presented by Levinson (1965) and Zubov (1970) based on the Legendre transformation. It may be possible to eliminate the deformation gradient from the Zubov's formulation, if the deformation gradient can be expressed only in terms of the first Piola–Kirchhoff stress; i.e., if there exists an inversion of constitutive law that defines the first Piola–Kirchhoff stress as a function of displacement gradient. Koiter (1976) proved that this inversion always exists, but *multi-valued* in usual case. From the mathematically rigorous definition, the lack of uniqueness means that there exists no inversion of constitutive law. However, we obey Koiter's terminology throughout this paper; i.e., we say that there exits a multi-valued inversion. He also presented the necessary condition for unique inversion such that the strain energy is a strictly convex and coercive function with respect to the displacement gradient. Even if this condition holds, it is not easy to express the complementary energy in an explicit form.

For many cases of small strain and finite deformation, the assumption of *semi-linear* material is practically acceptable, where the Biot stress is assumed to be a linear function of the right extensional strain. In the case of isotropic semi-linear material, Koiter (1976) presented the explicit expression of complementary energy function, which involves the Biot stress and the first Piola–Kirchhoff stress. Since these two stresses are related through the rotation, the subsidiary conditions of complementary energy principle may involve the unknown rotation. Allowing the multi-valued inversion of stress–strain relation, Mikkola (1989) presented the explicit form of complementary energy of trusses only in terms of internal force vectors. However, Mikkola's complementary energy function cannot be determined uniquely without information of unknown axial forces at the equilibrium state.

It should be noted that the stability of equilibrium state does not always mean the local minimum of the complementary energy (see, e.g., the counter example (Koiter, 1976)). Therefore, only the stationary principle of complementary energy has been discussed in the finite deformation theory for general structures such as elastic continuum, trusses, beams, etc.

Recently, the concepts of convex analysis have been applied to this problem; i.e., the complementary energy is defined by the (Fenchel's) conjugate transformation, instead of the Legendre transformation, of the strain energy function in terms of deformation gradient. In accordance with this definition, the complementary energy is now uniquely determined. However, the stationary principle using such a complementary energy is valid only if the strain energy is a convex function of deformation gradient. Usually, this condition is not satisfied, even in the case of linear material. Atai and Steigmann (1997) presented the relaxed strain energy of cable member as the lower convex envelope of strain energy of truss member, and formulated the minimum principle of complementary energy for cable networks by using the conjugate transformation. Unfortunately, they did not formulate the complementary energy function explicitly.

Observing that the total potential energy attains local minimum at the stable equilibrium state, it is natural to regard the minimization problem of total potential energy as a (finite or infinite dimensional) optimization problem. Under the assumption of small deformation and linear elastic material, the minimization problem of total potential energy can be formulated as a convex quadratic programming (QP) problem. Then, the minimization problem of complementary energy is obtained as the well-known Dorn's dual problem of QP, which is a separable dual problem; i.e., the primal and dual problems have no common variables. The validity of the obtained complementary principle as well as the zero duality gap can be shown by using the strong duality of QP. A general nonlinear programming problem does not have a separable dual problem, and there exists the duality gap between the primal and dual optimal objective values. This explains, from the view point of duality theory, the difficulty of formulating the extremum principle of complementary energy in the finite deformation theory. Recently, the extended concept of duality, which is referred to as *triality*, was presented by Gao (1997, 1999) and Gao and Strang (1989) for the extremum complementary energy principles of finite elasticity. Assuming nonsingularity of the second Piola–Kirchhoff stress, Gao's complementary energy is explicitly expressed in terms of both the first and the second Piola–Kirchhoff stresses. At the equilibrium state, this complementary energy attains the stationary point under the subsidiary constraints such that the first Piola–Kirchhoff stress satisfies the equilibrium equation.

In the authors' very recent paper (Kanno et al., 2002), it was shown for cable networks that the minimization problem of total potential energy can be reformulated as a second-order cone programming (SOCP) problem. SOCP is a class of convex programming including linear programming (LP) and QP, and is included in semi-definite programming (Vandenberghe and Boyd, 1996). SOCP has received increasing attention for its wide fields of application (Jarre et al., 1998; Vanderbei and Yurttan Benson, 1998; Ben-Tal and Nemirovski, 2001), and for the development of practical algorithms referred to as primal–dual interior-point methods (Monteiro and Tsuchiya, 2000).

SOCP is known to be a special case of LP over symmetric cone, and have a separable symmetric dual problem with zero duality gap (Ben-Tal and Nemirovski, 2001). Therefore, it is very natural to conjecture that the minimization problem of complementary energy is obtained as the dual SOCP problem, which involves no primal variables. In this paper, we propose a very simple and easy approach to establish the minimum complementary energy principle for cable networks; i.e., the minimization problem of complementary energy will be immediately derived by using the well-established duality theory of SOCP. In this approach, the ambiguity in the existence and uniqueness of inversion of constitutive law is successfully avoided. Moreover, the complementary energy function is obtained explicitly in a simple algebraic form. This advantage can be understood from the nice structure of corresponding Lagrangian suddle function.

This paper is organized as follows. In Section 2, we introduce the SOCP problem, and prepare the results about its strong duality. In Section 3, we present an SOCP problem in the standard form, which has the same optimizer as that of the minimization problem of total potential energy of cable networks. By using the framework of duality theory of SOCP, the separable dual problem to the minimization of total potential energy is obtained in Section 4. The strong duality and the optimality conditions of the obtained dual problem are investigated to guarantee that the optimal solution corresponds to the set of internal force vectors at the equilibrium state. Thus, the minimum principle of complementary energy is established in truly complementary form; i.e., the principle contains only stress components, and the sum of total potential energy and complementary energy becomes equal to zero at the equilibrium state. Section 5 is devoted to the discussion on the existence of solution. In Section 6, the concept of complementary work is revisited in order to investigate the physical meaning of the obtained complementary energy function. A simple cable network is examined in Section 7 to illustrate the results. Some remarks are given in Section 8, where our approach is compared with others in the literature from the unified view point of Lagrangian duality. It is interesting to see that the different approaches can be interpreted as the variety of Lagrangians, since the Lagrangian is a classical and rather familiar mathematical tool for researchers working on mechanics. It is also clarified how our approach can avoid the difficulties of coupling of stress and displacement and multi-valued inversion of constitutive law. Sections 9 and 10 are devoted to the proofs of lemmas and theorem in Sections 3 and 4, respectively.

2. Primal and dual pair of second-order cone programs

In this paper, all the vectors are assumed to be column vectors. To simplify the notation, however, define

$$(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}, \boldsymbol{b})^{\mathrm{T}} = (\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}})^{\mathrm{T}} \in \mathbf{R}^{n+m}$$

for vectors $\boldsymbol{a} \in \mathbf{R}^n$ and $\boldsymbol{b} \in \mathbf{R}^m$.

Let $\mathscr{K}(p)$ be the second-order cone in p-dimensional space defined as (Monteiro and Tsuchiya, 2000)

$$\mathscr{K}(p) = \{(s_0, \boldsymbol{s}_1) | s_0 \geqslant \|\boldsymbol{s}_1\|\},$$

where $s_0 \in \mathbf{R}$ and $\boldsymbol{s}_1 \in \mathbf{R}^{p-1}$. The notation $\|\boldsymbol{s}_1\|$ for $\boldsymbol{s}_1 \in \mathbf{R}^{p-1}$ denotes the Euclidean norm of the vector $\boldsymbol{s}_1$; i.e., $\|\boldsymbol{s}_1\| = (\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{s}_1)^{1/2}$. For a simple example of $p = 3$, the second-order cone is defined as

$$\mathscr{K}(3) = \left\{(x_0, x_1, x_2) | x_0 \geqslant \sqrt{x_1^2 + x_2^2}\right\},$$

where $\boldsymbol{s}_1 = (x_1, x_2)$ and $\boldsymbol{s} = (x_0, \boldsymbol{s}_1)$. It is easy to see that $\mathscr{K}(3)$ coincides with the surface and interior of a circular cone in three-dimensional space, which is as illustrated in Fig. 1.

$\mathscr{K}^*(p)$ denotes the dual cone of $\mathscr{K}(p)$, which is defined by using $\lambda_0 \in \mathbf{R}$ and $\boldsymbol{\lambda}_1 \in \mathbf{R}^{p-1}$ as

$$\mathscr{K}^*(p) = \{(\lambda_0, \boldsymbol{\lambda}_1) | \lambda_0 s_0 + \boldsymbol{\lambda}_1^{\mathrm{T}}\boldsymbol{s}_1 \geqslant 0, \quad \forall (s_0, \boldsymbol{s}_1) \in \mathscr{K}(p)\}.$$

Then $\mathscr{K}^*(p) = \mathscr{K}(p)$ holds, which is known as the self-duality (Monteiro and Tsuchiya, 2000; Ben-Tal and Nemirovski, 2001). It follows that

$$(\lambda_0, \boldsymbol{\lambda}_1) \in \mathscr{K}(p) \Longleftrightarrow \lambda_0 s_0 + \boldsymbol{\lambda}_1^{\mathrm{T}}\boldsymbol{s}_1 \geqslant 0, \quad \forall (s_0, \boldsymbol{s}_1) \in \mathscr{K}(p) \tag{1}$$

is satisfied. In the following, we often use the notation $s_0 \geqslant \|\boldsymbol{s}_1\|$ instead of $(s_0, \boldsymbol{s}_1) \in \mathscr{K}(p)$.

We consider the following SOCP problem:

$$\left.\begin{array}{lll} (P_{\mathrm{SOCP}}) : \min & \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} & \\ \quad\quad\quad\ \ \mathrm{s.t.} & \boldsymbol{A}_i\boldsymbol{x} = \boldsymbol{s}_i + \boldsymbol{e}_i, \quad s_{i0} \geqslant \|\boldsymbol{s}_{i1}\| \quad (i = 1, \ldots, k), \end{array}\right\} \tag{2}$$

where $\boldsymbol{x} \in \mathbf{R}^m$ and $\boldsymbol{s}_i = (s_{i0}, \boldsymbol{s}_{i1}) \in \mathbf{R}^{n_i}$ $(i = 1, \ldots, k)$ are variables, $\boldsymbol{d} \in \mathbf{R}^{\mathrm{m}}$ and $\boldsymbol{e}_i \in \mathbf{R}^{n_i}$ $(i = 1, \ldots, k)$ are constant vectors, and $\boldsymbol{A}_i \in \mathbf{R}^{n_i \times m}$ $(i = 1, \ldots, k)$ are constant matrices. The dimension of (P_{SOCP}) is determined by $(m, k, n_1, \ldots, n_k)$. The dual problem of (P_{SOCP}) is also an SOCP problem defined as

$$\left.\begin{array}{lll} (D_{\mathrm{SOCP}}) : \max & \displaystyle\sum_{i=1}^{k} \boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{z}_i & \\ \quad\quad\quad\ \ \mathrm{s.t.} & \displaystyle\sum_{i=1}^{k} \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{z}_i = \boldsymbol{d}, \quad z_{i0} \geqslant \|\boldsymbol{z}_{i1}\| \quad (i = 1, \ldots, k), \end{array}\right\} \tag{3}$$

where $\boldsymbol{z}_i = (z_{i0}, \boldsymbol{z}_{i1}) \in \mathbf{R}^{n_i}$ $(i = 1, \ldots, k)$ are variables. It is known that the problem dual to (D_{SOCP}) is (P_{SOCP}); i.e., the duality of SOCP is symmetric (Ben-Tal and Nemirovski, 2001).

To simplify the notation, define $N = \sum_{i=1}^{k} n_i$, $\boldsymbol{s} = (\boldsymbol{s}_1, \ldots, \boldsymbol{s}_k) \in \mathbf{R}^N$, and $\boldsymbol{z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k) \in \mathbf{R}^N$. We say that $(\boldsymbol{x}, \boldsymbol{s})$ and $\boldsymbol{z}$ are *feasible solutions* of (P_{SOCP}) and (D_{SOCP}), respectively, if they satisfy all the constraints of (P_{SOCP}) and (D_{SOCP}). Define the sets $\mathscr{F}^0(P_{\mathrm{SOCP}}) \subseteq \mathbf{R}^{m+N}$ and $\mathscr{F}^0(D_{\mathrm{SOCP}}) \subseteq \mathbf{R}^N$ by

$$\mathscr{F}^0(P_{\mathrm{SOCP}}) = \{(\boldsymbol{x}, \boldsymbol{s}) | \boldsymbol{A}_i\boldsymbol{x} = \boldsymbol{s}_i + \boldsymbol{e}_i, \quad s_{i0} > \|\boldsymbol{s}_i\| \quad (i = 1, \ldots, k)\},$$

$$\mathscr{F}^0(D_{\mathrm{SOCP}}) = \left\{\boldsymbol{z} \,\middle|\, \sum_{i=1}^{k} \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{z}_i = \boldsymbol{d}, \quad z_{i0} > \|\boldsymbol{z}_i\| \quad (i = 1, \ldots, k)\right\}.$$

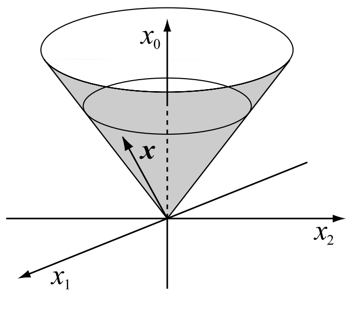

Fig. 1. Second-order cone in three-dimensional space.

$(\boldsymbol{x}, \boldsymbol{s}) \in \mathscr{F}^0(P_{\mathrm{SOCP}})$ and $\boldsymbol{z} \in \mathscr{F}^0(D_{\mathrm{SOCP}})$ are referred to as *interior-feasible solutions* of (P_{SOCP}) and (D_{SOCP}), respectively.

Assumption 2.1
(i) The m rows of $\boldsymbol{A}^{\mathrm{T}}$ are linearly independent, where $\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}_1^{\mathrm{T}}, \dots, \boldsymbol{A}_k^{\mathrm{T}}) \in \mathbf{R}^{m \times N}$.
(ii) $\mathscr{F}^0(P_{\mathrm{SOCP}}) \neq \emptyset$, $\mathscr{F}^0(D_{\mathrm{SOCP}}) \neq \emptyset$.

(P_{SOCP}) and (D_{SOCP}) are known to satisfy the following duality property:

Theorem 2.2 (Strong duality of SOCP). *Under Assumption* 2.1,

(i) (P_{SOCP}) *and* (D_{SOCP}) *have optimal solutions* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}})$ *and* $\bar{\boldsymbol{z}}$, *respectively, and*

$$\boldsymbol{d}^{\mathrm{T}} \bar{\boldsymbol{x}} = \sum_{i=1}^{k} \boldsymbol{e}_i^{\mathrm{T}} \bar{\boldsymbol{z}}_i. \tag{4}$$

(ii) *feasible solutions* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{s}})$ *and* $\bar{\boldsymbol{z}}$ *of* (P_{SOCP}) *and* (D_{SOCP}), *respectively, are optimal solutions* (a) *if and only if* (4) *is satisfied;* (b) *if and only if they satisfy*

$$\bar{\boldsymbol{s}}_i^{\mathrm{T}} \bar{\boldsymbol{z}}_i = 0 \quad (i = 1, \dots, k). \tag{5}$$

Proof. See Theorem 2.4.1 in Ben-Tal and Nemirovski (2001). □

Theorem 2.2 plays a fundamental role to establish the minimum principle of complementary energy in Section 4.

3. Minimum principle of total potential energy for cable networks

Consider a pin-jointed cable network in three-dimensional space. We assume a linear elastic material obeying Hooke's law. The network is discretized into members which connect pin-joints and supports.

Let n^{m} and n^{d} denote the numbers of members and unconstrained degrees of freedom, respectively. The reference state and actual equilibrium state are referred to as Γ^{I} and Γ^{II}, respectively. $\boldsymbol{x}^0 \in \mathbf{R}^{n^{\mathrm{d}}}$ and $\boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}$, respectively, denote the vectors of nodal coordinates at Γ^{I}, and nodal displacements at the deformed state corresponding to unconstrained degrees. We specify the external dead loads $\boldsymbol{f} \in \mathbf{R}^{n^{\mathrm{d}}}$ for unconstrained degrees.

Throughout this paper, we assume the following conditions, which are satisfied by most of actual cable networks:

Assumption 3.1. Suppose that the cable network $\mathscr{C}$ satisfies the conditions such that

(i) there is no more than one member which has the same adjacency;
(ii) each member connects two different nodes;
(iii) there is a path between any pair of nodes;
(iv) at least one freedom of displacement with respect to each coordinate x_1, x_2 and x_3 is constrained; i.e., $\mathscr{C}$ is not a free-body with respect to any direction.

Assumption 3.1 will be utilized to ensure the strong duality between the minimization problems of total potential energy and complementary energy (see Sections 4 and 5).

Let c_i and l_i^0 $(i = 1, \dots, n^{\rm m})$ denote the elongation regarded as the generalized strain, and the specified initial unstressed length of the ith member, respectively. The relation between c_i and $\boldsymbol{u}$ is written as

$$c_i + l_i^0 = \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\| \quad (i = 1, \dots, n^{\rm m}), \tag{6}$$

which is regarded as the geometrically exact compatibility condition. For each $i = 1, \dots, n^{\rm m}$, $\boldsymbol{B}_i \in \mathbf{R}^{3\times n^{\rm d}}$ is a constant matrix determined only by the connectivity of nodes and the ith member, each element of which is either $\{-1, 0, 1\}$. $\boldsymbol{b}_i \in \mathbf{R}^3$ is a constant vector that consists of the specified nodal coordinates of support if the ith member is connected to the support, otherwise $\boldsymbol{b}_i = \mathbf{0}$.

Let σ_i denote the axial force, which is a generalized stress. The cable member is assumed not to capable of transmitting the compression force; i.e., the relation between c_i and σ_i is written as

$$\sigma_i(c_i) = \begin{cases} k_i c_i & (c_i \geqslant 0), \\ 0 & (-l_i^0 \leqslant c_i < 0), \end{cases} \tag{7}$$

where $k_i > 0$ denotes the extensional stiffness of the ith member. Note that c_i defined by (6) satisfies $c_i \geqslant -l_i^0$ for any $\boldsymbol{u} \in \mathbf{R}^{n^{\rm d}}$. From (7), the strain energy w_i in terms of c_i is obtained as

$$w_i(c_i) = \int_0^{c_i} \sigma_i(c_i)\,{\rm d}c_i = \begin{cases} \dfrac{1}{2} k_i c_i^2 & (c_i \geqslant 0), \\ 0 & (-l_i^0 \leqslant c_i < 0). \end{cases} \tag{8}$$

By using (6) and (8), the problem of minimum total potential energy is formulated as

$$\left.\begin{array}{lll} (\Pi): \min & \Pi(\boldsymbol{c}, \boldsymbol{u}) = \sum_{i=1}^{n^{\rm m}} w_i(c_i) - \boldsymbol{f}^{\rm T}\boldsymbol{u} & \\ \text{s.t.} & c_i = \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\| - l_i^0 & (i = 1, \dots, n^{\rm m}), \end{array}\right\} \tag{9}$$

where independent variables are $\boldsymbol{u} \in \mathbf{R}^{n^{\rm d}}$. Let $\boldsymbol{c}^{\rm II} = (c_i^{\rm II}) \in \mathbf{R}^{n^{\rm m}}$ and $\boldsymbol{u}^{\rm II} \in \mathbf{R}^{n^{\rm d}}$, respectively, denote the vectors of member elongation and nodal displacements at $\Gamma^{\rm II}$. The principle of minimum total potential energy states that $(\boldsymbol{c}^{\rm II}, \boldsymbol{u}^{\rm II})$ is an optimal solution of (Π).

Notice here that (Π) is a nonconvex optimization problem, since c_i defined by (6) is a nonconvex function of $\boldsymbol{u}$. This implies that the classical Lagrangian dual problem of (Π) contains unknown $\boldsymbol{u}$, and does not satisfy the strong duality. Therefore, it is not straightforward to establish the dual minimum principle to (Π) without duality gap only in terms of the stress components. See Section 8 for more details. This difficulty motivates us to investigate the following convex problem:

$$\left.\begin{array}{lll} (P): \min & \phi^P(\boldsymbol{y}, \boldsymbol{u}) = \sum_{i=1}^{n^{\rm m}} \dfrac{1}{2} k_i y_i^2 - \boldsymbol{f}^{\rm T}\boldsymbol{u} & \\ \text{s.t.} & y_i + l_i^0 \geqslant \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\| & (i = 1, \dots, n^{\rm m}), \end{array}\right\} \tag{10}$$

where $\boldsymbol{y} = (y_i) \in \mathbf{R}^{n^{\rm m}}$ and $\boldsymbol{u} \in \mathbf{R}^{n^{\rm d}}$ are independent variables. The following lemma gives the relation between optimal solutions of (Π) and (P):

Lemma 3.2 (Relation between (Π) and (P)).
(i) $(\boldsymbol{c}^{\rm II}, \boldsymbol{u}^{\rm II})$ *satisfying* (6) *is an optimal solution of* (Π) *if and only if* $(\bar{\boldsymbol{y}}, \boldsymbol{u}^{\rm II})$ *defined as*

$$\bar{y}_i = \begin{cases} c_i^{\rm II} & (c_i^{\rm II} \geqslant 0) \\ 0 & (-l_i^0 \leqslant c_i^{\rm II} < 0) \end{cases} \quad (i = 1, \dots, n^{\rm m}) \tag{11}$$

is an optimal solution of (P).
(ii) *If* $(\boldsymbol{c}^{\rm II}, \boldsymbol{u}^{\rm II})$ *and* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$ *are optimal solutions of* (Π) *and* (P), *respectively, then* $\Pi(\boldsymbol{c}^{\rm II}, \boldsymbol{u}^{\rm II}) = \phi^P(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$.

Proof. See Kanno et al. (2002) (Section 3). □

Lemma 3.2 implies that $\boldsymbol{u}^{\rm II}$ can be obtained by solving (P) instead of (Π). It has been shown that (P) is easily solved compared with (Π). Indeed, the primal–dual interior-point method is a very effective algorithm with polynomial-time convergence to obtain an optimal solution of (P). See Kanno et al. (2002) for more details.

By using the convexity of ϕ^P and after simple algebraic manipulation, (P) can be reduced to an SOCP problem. Consider the following problem:

$$\left.\begin{aligned} (P_{\rm S}):\ \min \quad & \phi_{\rm S}^P(\boldsymbol{u},\boldsymbol{t}) = \sum_{i=1}^{n^{\rm m}} t_i - \boldsymbol{f}^{\rm T}\boldsymbol{u} \\ \text{s.t.} \quad & \frac{t_i}{2k_i} + 1 \geqslant \left\| \begin{pmatrix} \dfrac{t_i}{2k_i} - 1 \\ y_i \end{pmatrix} \right\|, \ y_i + l_i^0 \geqslant \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\| \quad (i = 1,\dots,n^{\rm m}), \end{aligned}\right\} \tag{12}$$

where independent variables are $\boldsymbol{y} \in \mathbf{R}^{n^{\rm m}}$, $\boldsymbol{u} \in \mathbf{R}^{n^{\rm d}}$, and $\boldsymbol{t} = (t_i) \in \mathbf{R}^{n^{\rm m}}$. The following lemma implies that (P) and $(P_{\rm S})$ are equivalent:

Lemma 3.3 (Relation between (P) and $(P_{\rm S})$).

(i) $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$ *is an optimal solution of* (P) *if and only if* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{t}})$ *satisfying*

$$\bar{t}_i = \frac{k_i}{2}\bar{y}_i^2 \quad (i = 1,\dots,n^{\rm m}) \tag{13}$$

is an optimal solution of $(P_{\rm S})$.

(ii) *If* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$ *and* $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{t}})$ *are optimal solutions of* (P) *and* $(P_{\rm S})$, *respectively, then* $\phi^P(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}) = \phi_{\rm S}^P(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{t}})$.

Proof. See Section 9. □

Note that $(P_{\rm S})$ has the linear objective function, and the feasible set of $(P_{\rm S})$ is represented by the $2n^{\rm m}$ second-order cones; i.e., $(P_{\rm S})$ is an SOCP problem. Indeed, it will be seen that $(P_{\rm S})$ can be embedded into $(P_{\rm SOCP})$ in the following section.

4. Minimum principle of complementary energy

In order to formulate the dual problem of (Π), we first investigate the dual problem of $(P_{\rm S})$, which is referred to as $(D_{\rm S})$. Because $(P_{\rm S})$ is an SOCP problem, $(D_{\rm S})$ can be obtained by simple algebraic manipulation by using the framework of SOCP duality theory introduced in Section 2. To this end, it is required to express $(P_{\rm S})$ in the form of $(P_{\rm SOCP})$. Recall that $(P_{\rm SOCP})$ and $(D_{\rm SOCP})$ have no common variables; i.e., an SOCP problem has the separable dual problem, which is rather unusual property for nonlinear optimization problem. This special structure, as well as the strong duality (Theorem 2.2), of SOCP is an indispensable property to the truly complementary energy principle, otherwise the dual problem contains the unknown displacement components such as rotations.

For the dimensions of $(P_{\rm SOCP})$, assign

$$\begin{aligned} & m = 2n^{\rm m} + n^{\rm d}, \quad k = 2n^{\rm m}, \\ & n_1 = \cdots = n_{n^{\rm m}} = 4, \quad n_{n^{\rm m}+1} = \cdots = n_{2n^{\rm m}} = 3. \end{aligned} \tag{14}$$

Let

$$\boldsymbol{d} = (\mathbf{1}, \mathbf{0}, -\boldsymbol{f}\ ) \in \mathbf{R}^{2n^{\rm m}+n^{\rm d}}, \quad \boldsymbol{x} = (\boldsymbol{t}, \boldsymbol{y}, \boldsymbol{u}) \in \mathbf{R}^{2n^{\rm m}+n^{\rm d}}, \tag{15}$$

where $\mathbf{0} = (0, \dots, 0) \in \mathbf{R}^{n^{\mathrm{m}}}$ and $\mathbf{1} = (1, \dots, 1) \in \mathbf{R}^{n^{\mathrm{m}}}$. Consequently, we can see $\phi_{\mathrm{S}}^{P}(\boldsymbol{u}, \boldsymbol{t}) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}$, which implies that the objective function of (P_{S}) is embedded into that of (P_{SOCP}).

Next, consider the constraints of (P_{S}). Define $\boldsymbol{\theta}^i = (\theta^i_j) \in \mathbf{R}^{n^{\mathrm{m}}}$ $(i = 1, \dots, n^{\mathrm{m}})$ as

$$\theta^i_j = \begin{cases} 1 & (j = i), \\ 0 & (\text{otherwise}). \end{cases}$$

Assigning $\boldsymbol{s}_i$ $(i = 1, \dots, 2n^{\mathrm{m}})$ as

$$\boldsymbol{s}_i = (y_i + l_i^0, \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i)^{\mathrm{T}}, \quad \boldsymbol{s}_{n^{\mathrm{m}}+i} = \left(\frac{t_i}{2k_i} + 1, \frac{t_i}{2k_i} - 1, y_i \right)^{\mathrm{T}} \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{16}$$

$\boldsymbol{x}$ and $\boldsymbol{s}_i$ satisfy the linear equality constraints $\boldsymbol{A}_i\boldsymbol{x} = \boldsymbol{s}_i + \boldsymbol{e}_i$ in (P_{SOCP}) with

$$\boldsymbol{A}_i = \begin{bmatrix} \mathbf{0}^{\mathrm{T}} & \boldsymbol{\theta}^{i\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_i \end{bmatrix}, \quad \boldsymbol{A}_{n^{\mathrm{m}}+i} = \begin{bmatrix} \frac{1}{2k_i}\boldsymbol{\theta}^{i\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \frac{1}{2k_i}\boldsymbol{\theta}^{i\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} & \boldsymbol{\theta}^{i\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{bmatrix} \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{17}$$

$$\boldsymbol{e}_i = (-l_i^0, -\boldsymbol{B}_i\boldsymbol{x}^0 + \boldsymbol{b}_i)^{\mathrm{T}}, \quad \boldsymbol{e}_{n^{\mathrm{m}}+i} = (-1, 1, 0)^{\mathrm{T}} \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{18}$$

Accordingly, we see for $i = 1, \dots, n^{\mathrm{m}}$ that the inequality constraint $s_{i0} \geqslant \|\boldsymbol{s}_{i1}\|$ in (P_{SOCP}) corresponds to the first constraint in (P_{S}). For $i = n^{\mathrm{m}} + 1, \dots, 2n^{\mathrm{m}}$, $s_{i0} \geqslant \|\boldsymbol{s}_{i1}\|$ in (P_{SOCP}) corresponds to the second constraint in (P_{S}). Thus, (P_{S}) is embedded into (P_{SOCP}).

In the following, the dual problem of (P_{S}), which is referred to as (D_{S}), will be derived from (D_{SOCP}) along with the definitions of (14), (15), (17) and (18). Assigning $\boldsymbol{z}_i$ in (D_{SOCP}) as

$$\boldsymbol{z}_i = (q_i, \boldsymbol{v}_i)^{\mathrm{T}} \in \mathbf{R}^4, \quad \boldsymbol{z}_{n^{\mathrm{m}}+i} = \boldsymbol{\xi}_i = (\xi_{i0}, \xi_{i1}, \xi_{i2})^{\mathrm{T}} \in \mathbf{R}^3 \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{19}$$

(17)–(19) lead to

$$\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{z}_i = \begin{pmatrix} \mathbf{0} \\ q_i\boldsymbol{\theta}^i \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{v}_i \end{pmatrix} \in \mathbf{R}^{2n^{\mathrm{m}}+n^{\mathrm{d}}}, \quad \boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{z}_i = -l_i^0 q_i - (\boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{b}_i)^{\mathrm{T}}\boldsymbol{v}_i \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{20}$$

$$\boldsymbol{A}_{n^{\mathrm{m}}+i}^{\mathrm{T}}\boldsymbol{z}_{n^{\mathrm{m}}+i} = \begin{pmatrix} \frac{\xi_{i0} + \xi_{i1}}{2k_i}\boldsymbol{\theta}^i \\ \xi_{i2}\boldsymbol{\theta}^i \\ \mathbf{0} \end{pmatrix} \in \mathbf{R}^{2n^{\mathrm{m}}+n^{\mathrm{d}}}, \quad \boldsymbol{e}_{n^{\mathrm{m}}+i}^{\mathrm{T}}\boldsymbol{z}_{n^{\mathrm{m}}+i} = -\xi_{i0} + \xi_{i1} \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{21}$$

It follows from (15), (20) and (21) that the equality constraints $\sum_{i=1}^{k} \boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{z}_i = \boldsymbol{d}$ in (D_{SOCP}) are reduced to

$$\sum_{i=1}^{n^{\mathrm{m}}} \begin{pmatrix} \frac{\xi_{i0} + \xi_{i1}}{2k_i}\boldsymbol{\theta}^i \\ (q_i + \xi_{i2})\boldsymbol{\theta}^i \\ \boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{v}_i \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ -\boldsymbol{f} \end{pmatrix}. \tag{22}$$

By using (19), we see that the inequality constraints $z_{i0} \geqslant \|\boldsymbol{z}_{i1}\|$ $(i = 1, \dots, 2n^{\mathrm{m}})$ in (D_{SOCP}) are reduced to

$$q_i \geqslant \|\boldsymbol{v}_i\|, \quad \xi_{i0} \geqslant \|(\xi_{i1}, \xi_{i2})^{\mathrm{T}}\| \quad (i = 1, \dots, n^{\mathrm{m}}). \tag{23}$$

By substituting (20)–(23) into $(D_{\rm SOCP})$, the dual problem of $(P_{\rm S})$ is obtained as

$$\left.\begin{array}{ll}(D_{\rm S}): \max & \phi_{\rm S}^{D}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{\xi}) = \sum_{i=1}^{n^{\rm m}}(-\xi_{i0}+\xi_{i1}) - \sum_{i=1}^{n^{\rm m}} l_i^0 q_i - \sum_{i=1}^{n^{\rm m}}(\boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{b}_i)^{\rm T}\boldsymbol{v}_i \\ \quad\text{s.t.} & \xi_{i0}+\xi_{i1} = 2k_i, \xi_{i0} \geqslant \|(\xi_{i1},\xi_{i2})^{\rm T}\| \quad (i=1,\dots,n^{\rm m}), \\ & q_i + \xi_{i2} = 0, \ q_i \geqslant \|\boldsymbol{v}_i\| \quad (i=1,\dots,n^{\rm m}), \\ & \sum_{i=1}^{n^{\rm m}} \boldsymbol{B}_i^{\rm T}\boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0}, \end{array}\right\} \tag{24}$$

where independent variables are $\boldsymbol{q}_i = (q_i) \in \mathbf{R}^{n^{\rm m}}$, $\boldsymbol{v} = (\boldsymbol{v}_1,\dots,\boldsymbol{v}_{n^{\rm m}}) \in \mathbf{R}^{3n^{\rm m}}$, and $\boldsymbol{\xi} = (\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_{n^{\rm m}}) \in \mathbf{R}^{3n^{\rm m}}$. Define the function $w_i^{\rm C}$ on $\boldsymbol{v}_i \in \mathbf{R}^3$ by

$$w_i^{\rm C}(\boldsymbol{v}_i) = \frac{\boldsymbol{v}_i^{\rm T}\boldsymbol{v}_i}{2k_i}, \tag{25}$$

which coincides with the complementary strain energy in small deformation theory. Consider the following problem:

$$\left.\begin{array}{ll}(\Pi^{\rm C}): \min & \Pi^{\rm C}(\boldsymbol{v}) = \sum_{i=1}^{n^{\rm m}} w_i^{\rm C}(\boldsymbol{v}_i) + \sum_{i=1}^{n^{\rm m}} l_i^0\|\boldsymbol{v}_i\| + \sum_{i=1}^{n^{\rm m}}(\boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{b}_i)^{\rm T}\boldsymbol{v}_i \\ \quad\text{s.t.} & \sum_{i=1}^{n^{\rm m}} \boldsymbol{B}_i^{\rm T}\boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0}, \end{array}\right\} \tag{26}$$

where independent variables are $\boldsymbol{v} = (\boldsymbol{v}_1,\dots,\boldsymbol{v}_{n^{\rm m}}) \in \mathbf{R}^{3n^{\rm m}}$. We can show the following lemma relating optimal solutions of $(D_{\rm S})$ and $(\Pi^{\rm C})$:

Lemma 4.1 (Relation between $(D_{\rm S})$ and $(\Pi^{\rm C})$).

(i) *$\boldsymbol{v}^{\rm II}$ is an optimal solution of $(\Pi^{\rm C})$ if and only if $(\bar{\boldsymbol{q}},\boldsymbol{v}^{\rm II},\bar{\boldsymbol{\xi}})$ satisfying*

$$\bar{\xi}_{i0} = \frac{\bar{q}_i^2}{4k_i} + k_i, \quad \bar{\xi}_{i1} = -\frac{\bar{q}_i^2}{4k_i} + k_i, \quad \bar{\xi}_{i2} = -\bar{q}_i = -\|\boldsymbol{v}_i^{\rm II}\| \quad (i=1,\dots,n^{\rm m})$$

is an optimal solution of $(D_{\rm S})$.

(ii) *If $\boldsymbol{v}^{\rm II}$ and $(\bar{\boldsymbol{q}},\bar{\boldsymbol{v}},\bar{\boldsymbol{\xi}})$ are optimal solutions of $(\Pi^{\rm C})$ and $({\rm D_S})$, respectively, then $\Pi^{\rm C}(\boldsymbol{v}^{\rm II}) = -\phi_{\rm S}^{D}(\bar{\boldsymbol{q}},\bar{\boldsymbol{v}},\bar{\boldsymbol{\xi}})$.*

Proof. See Section 9. □

Notice again that $(P_{\rm S})$ and $(D_{\rm S})$ are primal–dual pair of SOCP problems satisfying the strong duality (Theorem 2.2). Lemmas 3.2 and 3.3 imply the equivalence between $(P_{\rm S})$ and (Π), and Lemma 4.1 shows the equivalence between $(D_{\rm S})$ and $(\Pi^{\rm C})$. Therefore, the strong duality between (Π) and $(\Pi^{\rm C})$ is immediately obtained from the duality theorem of SOCP. Define the set $\mathscr{F}^0(\Pi^{\rm C}) \subseteq \mathbf{R}^{3n^{\rm m}}$ as

$$\mathscr{F}^0(\Pi^{\rm C}) = \left\{ (\boldsymbol{v}_1,\dots,\boldsymbol{v}_{n^{\rm m}}) \in \mathbf{R}^{3n^{\rm m}} \,\middle|\, \sum_{i=1}^{n^{\rm m}} \boldsymbol{B}_i^{\rm T}\boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0} \right\}.$$

The following lemma should be prepared:

Lemma 4.2. *If Assumption* 3.1 *is satisfied, then*

(i) *the $n^{\rm d}$ rows of $\boldsymbol{B}^{\rm T}$ are linearly independent, where $\boldsymbol{B}^{\rm T} = (\boldsymbol{B}_1^{\rm T},\dots,\boldsymbol{B}_{n^{\rm m}}^{\rm T}) \in \mathbf{R}^{n^{\rm d}\times 3n^{\rm m}}$;*

(ii) *$\mathscr{F}^0(\Pi^{\rm C}) \neq \emptyset$ for any $\boldsymbol{f} \in \mathbf{R}^{n^{\rm d}}$.*

Proof
(i) See Section 5.
(ii) This assertion follows from Lemma 4.2(i) immediately. □

Lemma 4.2 is utilized to show that Assumption 3.1 guarantees Assumption 2.1, which is a sufficient condition for strong duality of primal–dual pair of SOCP problems. The following theorem is the main result of this paper:

Theorem 4.3 (Strong duality of (Π) and (Π^{C})). *Under Assumption* 3.1.

(i) *(Π) and (Π^{C}) have optimal solutions ($\boldsymbol{c}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}}$) and $\boldsymbol{v}^{\mathrm{II}}$, respectively, and* $\Pi(\boldsymbol{c}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}}) = -\Pi^{\mathrm{C}}(\boldsymbol{v}^{\mathrm{II}})$.
(ii) *($\boldsymbol{c}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}}$) and $\boldsymbol{v}^{\mathrm{II}}$ are optimal solutions of (Π) and (Π^{C}), respectively, if and only if there exists a vector* $(\boldsymbol{h}_1^{\mathrm{II}}, \ldots, \boldsymbol{h}_{n^{\mathrm{m}}}^{\mathrm{II}}) \in \mathbf{R}^{3n^{\mathrm{m}}}$ *satisfying*

$$\boldsymbol{h}_i^{\mathrm{II}} = \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}^{\mathrm{II}}) - \boldsymbol{b}_i \quad (i = 1, \ldots, n^{\mathrm{m}}), \tag{27}$$

$$c_i^{\mathrm{II}} = \|\boldsymbol{h}_i^{\mathrm{II}}\| - l_i^0 \quad (i = 1, \ldots, n^{\mathrm{m}}), \tag{28}$$

$$\boldsymbol{v}_i^{\mathrm{II}} = \begin{cases} -\sigma_i(c_i^{\mathrm{II}}) \frac{\boldsymbol{h}_i^{\mathrm{II}}}{\|\boldsymbol{h}_i^{\mathrm{II}}\|} & (\boldsymbol{h}_i^{\mathrm{II}} \neq \mathbf{0}) \\ \mathbf{0} & (\boldsymbol{h}_i^{\mathrm{II}} = \mathbf{0}) \end{cases} \quad (i = 1, \ldots, n^{\mathrm{m}}), \tag{29}$$

$$\sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i^{\mathrm{T}} \boldsymbol{v}_i^{\mathrm{II}} + \boldsymbol{f} = \mathbf{0}. \tag{30}$$

Proof. See Section 10. □

Theorem 4.3(ii) and the principle of minimum potential energy imply that c_i^{II} and $\boldsymbol{u}^{\mathrm{II}}$ satisfying (27)–(30) correspond to member elongation and vector of nodal displacements at Γ^{II}. Accordingly, for $i = 1, \ldots, n^{\mathrm{m}}$, $\boldsymbol{h}_i^{\mathrm{II}} \in \mathbf{R}^3$ defined by (27) corresponds to the vector with the same direction and length as the ith member at Γ^{II}. It follows from (29) that $\boldsymbol{h}_i^{\mathrm{II}}$ and $-\boldsymbol{v}_i^{\mathrm{II}}$ have the same direction as illustrated in Fig. 2, and $\|\boldsymbol{v}_i^{\mathrm{II}}\|$ is equivalent to the axial force; i.e., $-\boldsymbol{v}_i^{\mathrm{II}}$ corresponds to the internal force vector of the ith member at Γ^{II}. The condition (30) corresponds to the equilibrium equations in terms of internal forces. Hence, the complementary energy principle now can be stated such that the internal forces $\boldsymbol{v}_1^{\mathrm{II}}, \ldots, \boldsymbol{v}_{n^{\mathrm{m}}}^{\mathrm{II}}$ are obtained by minimizing $\Pi^{\mathrm{C}}(\boldsymbol{v})$ over the constraints of equilibrium equations. It should be emphasized that $\Pi^{\mathrm{C}}(\boldsymbol{v})$ is

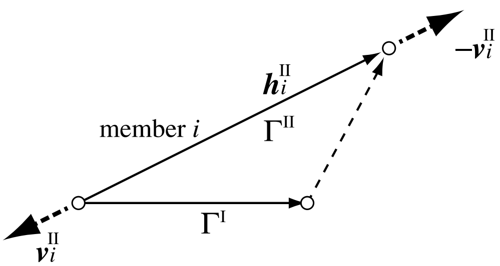

Fig. 2. Physical interpretation of $\boldsymbol{v}_i^{\mathrm{II}}$.

determined uniquely for any $\boldsymbol{v} \in \mathbf{R}^{3n^{\mathrm{m}}}$. Moreover, Theorem 4.3(i) guarantees the zero duality gap between (Π) and (Π^{C}).

5. Existence and uniqueness of the solution

Lemma 4.2 and Theorem 4.3 imply that Assumption 3.1 is a sufficient condition for existence of solutions of (Π) and (Π^{C}). In this section, the existence and uniqueness of solution are investigated based on the framework of graph theory.

Definition 5.1. For a cable network $\mathscr{C}$,

(i) the unconstrained cable network $\mathscr{C}^*$ is uniquely defined by removing all the constraints of displacements from $\mathscr{C}$. Conversely, $\mathscr{C}$ is obtained by adding constraints at all the nodes in the set $\mathscr{J}^k$ $(k = 1, 2, 3)$ of $\mathscr{C}^*$ in the direction of x_k;
(ii) the directed graph $\mathscr{G}(\mathscr{C})$ is defined by regarding each node and member of $\mathscr{C}^*$ as vertex and edge with any direction, respectively.

Note that $|\mathscr{J}^k| = n^{\mathrm{n}} - n^{\mathrm{d}}_k$, where n^{n} and n^{d}_k denote the numbers of nodes and degrees of freedom of $\mathscr{C}$ in the direction x_k $(k = 1, 2, 3)$, respectively. Assumption 3.1(iv) implies $|\mathscr{J}^k| \geqslant 1$. It follows from Definition 5.1 that $\mathscr{C}^*$ has no support, and the number of degrees of freedom is $3n^{\mathrm{n}}$. By using the terminologies in graph theory, Assumption 3.1(i)–(iii) can be alternatively written as (i) $\mathscr{G}(\mathscr{C})$ has no multiple edges, (ii) $\mathscr{G}(\mathscr{C})$ has no loop, (iii) $\mathscr{G}(\mathscr{C})$ is connected; i.e., Assumption 3.1(i)–(iii) implies that $\mathscr{G}(\mathscr{C})$ is a connected simple directed graph (see, e.g., Wilson, 1985 for basic background of graph theory).

Proof of Lemma 4.2 (*i*). Let $\boldsymbol{D} \in \mathbf{R}^{n^{\mathrm{n}} \times n^{\mathrm{m}}}$ denote the incidence matrix of $\mathscr{G}(\mathscr{C})$. $\boldsymbol{v}^k \in \mathbf{R}^{n^{\mathrm{m}}}$ $(k = 1, 2, 3)$ denote the vector composed of the components of $\boldsymbol{v} \in \mathbf{R}^{3n^{\mathrm{m}}}$ in x_k-direction. Similarly, the external load vector $\boldsymbol{f} \in \mathbf{R}^{n^{\mathrm{d}}}$ is divided into the set of vectors $\boldsymbol{f}^k \in \mathbf{R}^{n^{\mathrm{d}}_k}$. Define the matrices $\boldsymbol{D}^k \in \mathbf{R}^{n^{\mathrm{d}}_k \times n^{\mathrm{m}}}$ by removing the jth row in $\mathscr{J}^k$ from $\boldsymbol{D}$, and then we see that $\mathscr{F}^0(\Pi^{\mathrm{C}})$ can be rewritten as

$$\mathscr{F}^0(\Pi^{\mathrm{C}}) = \{(\boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^3) | \boldsymbol{D}^k \boldsymbol{v}^k + \boldsymbol{f}^k = \boldsymbol{0} \ (k = 1, 2, 3)\}. \tag{31}$$

Alternatively, as a result of appropriate permutations of columns and rows of $\boldsymbol{B}^{\mathrm{T}}$, the matrix

$$\begin{bmatrix} \boldsymbol{D}^1 & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D}^2 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{D}^3 \end{bmatrix}$$

can be obtained. Therefore, we have only to show $\boldsymbol{D}^k = n^{\mathrm{d}}_k$ $(k = 1, 2, 3)$.

There exists $j_1 \in \mathscr{J}^k$ because $|\mathscr{J}^k| \geqslant 1$. Let $\boldsymbol{D}(j_1) \in \mathbf{R}^{(n^{\mathrm{n}}-1) \times n^{\mathrm{m}}}$ denote the matrix obtained by removing the j_1th row from $\boldsymbol{D}$, which is referred to as a *truncated incidence matrix* in the graph theory. For a connected simple directed graph, it is well known that $\boldsymbol{D}(j_1) = n^{\mathrm{n}} - 1$ (Chvátal, 1983, Chapter 19). Since any row of $\boldsymbol{D}^k$ is a row of $\boldsymbol{D}(j_1)$, all rows of $\boldsymbol{D}^k$ are linearly independent. □

As a consequence of Lemma 4.2 and Theorem 4.3, Assumption 3.1 guarantees that both (Π) and (Π^{C}) have optimal solutions for any $\boldsymbol{f} \in \mathbf{R}^{n^{\mathrm{d}}}$. Assuming now that $\mathscr{C}$ does not satisfy Assumption 3.1(iv); i.e., $\boldsymbol{D}^k = \boldsymbol{D}$ $(\exists k)$, we see $\operatorname{rank} \boldsymbol{D} = n^{\mathrm{n}} - 1$ (Chvátal, 1983), and hence $\mathscr{F}^0(\Pi^{\mathrm{C}}) = \emptyset$ for a certain $\boldsymbol{f} \in \mathbf{R}^{n^{\mathrm{d}}}$. Accordingly, Assumption 3.1(iv) is a necessary condition for existence of solutions of (Π) and (Π^{C}).

The remainder of this section is devoted to the discussion of uniqueness of solution.

Lemma 5.2 (Uniqueness of $\boldsymbol{v}^{\mathrm{II}}$). *Under Assumption* 3.1, $\boldsymbol{v}^{\mathrm{II}} = (\boldsymbol{v}_1^{\mathrm{II}}, \dots, \boldsymbol{v}_{n^{\mathrm{m}}}^{\mathrm{II}})$ *exists uniquely.*

Proof. For $i = 1, \dots, n^{\mathrm{m}}$, $\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{v}_i$ and $\|\boldsymbol{v}_i\|$ are strictly convex functions of $\boldsymbol{v}_i \in \mathbf{R}^3$. Accordingly, (Π^{C}) is a minimization problem of a strictly convex function $\Pi^{\mathrm{C}}(\boldsymbol{v})$ over linear equality constraints, from which and Theorem 4.3 it follows that (Π^{C}) has the unique solution $\boldsymbol{v}^{\mathrm{II}}$. □

Suppose that the ith member is in tensile state; i.e., $\boldsymbol{v}_i^{\mathrm{II}} \neq \boldsymbol{0}$. It follows from (7) that $\sigma_i^{-1}(\|\boldsymbol{v}_i^{\mathrm{II}}\|)$ exists uniquely. From (29), we have $\|\boldsymbol{v}_i^{\mathrm{II}}\| = \sigma_i(c_i^{\mathrm{II}})$, which implies $c_i^{\mathrm{II}} = \sigma_i^{-1}(\|\boldsymbol{v}_i^{\mathrm{II}}\|)$. By using (28), (29) can be reduced to

$$\boldsymbol{h}_i^{\mathrm{II}} = -\left[\sigma_i^{-1}(\|\boldsymbol{v}_i^{\mathrm{II}}\|) + l_i^0\right]\frac{\boldsymbol{v}_i^{\mathrm{II}}}{\|\boldsymbol{v}_i^{\mathrm{II}}\|}, \tag{32}$$

and we see that $\boldsymbol{h}_i^{\mathrm{II}}$ exists uniquely. Hence, $\boldsymbol{v}_i^{\mathrm{II}} \neq \boldsymbol{0}$ is a sufficient condition for the uniqueness of $\boldsymbol{h}_i^{\mathrm{II}}$. On the contrary, suppose $\boldsymbol{v}_i^{\mathrm{II}} = \boldsymbol{0}$. Then, the inversion of (29) is not unique, which implies that the deformation $\boldsymbol{u}^{\mathrm{II}}$ is not necessarily unique. Define $\overline{\mathscr{C}}$ by removing all the members satisfying $\boldsymbol{v}_i^{\mathrm{II}} = \boldsymbol{0}$ from $\mathscr{C}$. In accordance with Definition 5.1, the graph $\mathscr{G}(\overline{\mathscr{C}})$ is defined for $\overline{\mathscr{C}}$.

Lemma 5.3 (Uniqueness of $\boldsymbol{u}^{\mathrm{II}}$). *If*

(i) $\overline{\mathscr{C}}$ *satisfies Assumption* 3.1*(iv), and*
(ii) $\mathscr{G}(\overline{\mathscr{C}})$ *has a subgraph which is a spanning tree of* $\mathscr{G}(\mathscr{C})$,

then $\boldsymbol{u}^{\mathrm{II}}$ *of* $\mathscr{C}$ *is unique.*

Proof. Eq. (32) implies that $\boldsymbol{h}_i^{\mathrm{II}}$ exists uniquely for any member of $\overline{\mathscr{C}}$, from which and Lemma 5.3(i) it follows that the deformation of $\overline{\mathscr{C}}$ is unique. From Lemma 5.3(ii), $\overline{\mathscr{C}}$ has all the nodes of $\mathscr{C}$, which concludes the proof. □

Volokh and Vilnay (2000) showed that the tangent stiffness matrix is positive definite, if all the members are in tension. The results of Lemmas 5.2 and 5.3 are also seen in Atai and Steigmann (1997). However, to the authors' knowledge, no proof for finite deformation has been published based on the graph theory.

6. Physical interpretation of the complementary energy function

In this section, we investigate the physical meaning of the obtained complementary energy function in (Π^{C}). Let $\boldsymbol{x}^0 \in \mathbf{R}^{n^{\mathrm{d}}}$ and $\boldsymbol{b}_i^0 \in \mathbf{R}^3$ $(i = 1, \dots, n^{\mathrm{m}})$ denote the vectors of nodal coordinates corresponding to unconstrained degrees and support degrees, respectively. Similarly, recall that the vectors at the deformed state are denoted by $\boldsymbol{x}^0 + \boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}$ and $\boldsymbol{b}_i \in \mathbf{R}^3$ $(i = 1, \dots, n^{\mathrm{m}})$. Note that $\boldsymbol{x}^0$, $\boldsymbol{b}_i^0$, and $\boldsymbol{b}_i$ are specified. For $i = 1, \dots, n^{\mathrm{m}}$, $\hat{\boldsymbol{u}}_i = \boldsymbol{b}_i - \boldsymbol{b}_i^0 \in \mathbf{R}^3$ denotes the prescribed displacements of the support connected to the ith member.

Under the assumption of small strain and small rotation, the complementary energy function $\Pi_{\mathrm{LIN}}^{\mathrm{C}}$ for trusses is well known to be given by

$$\Pi_{\mathrm{LIN}}^{\mathrm{C}}(\boldsymbol{v}) = \sum_{i=1}^{n^{\mathrm{m}}} w_i^{\mathrm{C}}(\boldsymbol{v}_i) - \sum_{i=1}^{n^{\mathrm{m}}} \hat{\boldsymbol{u}}_i^{\mathrm{T}}\boldsymbol{v}_i, \tag{33}$$

where w_i^{C} defined by (25) is the complementary strain energy of the ith member. On the other hand, allowing finite deformation, we have shown in Section 4 that the complementary energy for cable networks can be written as

$$\Pi^{\mathrm{C}}(\boldsymbol{v}) = \sum_{i=1}^{n^{\mathrm{m}}} \tilde{w}_i^{\mathrm{C}}(\boldsymbol{v}_i) - \sum_{i=1}^{n^{\mathrm{m}}} \hat{\boldsymbol{u}}_i^{\mathrm{T}} \boldsymbol{v}_i, \tag{34}$$

where

$$\tilde{w}_i^{\mathrm{C}}(\boldsymbol{v}_i) = w_i^{\mathrm{C}}(\boldsymbol{v}_i) + l_i^0 \|\boldsymbol{v}_i\| - (\boldsymbol{B}_i \boldsymbol{x}^0 - \boldsymbol{b}_i^0)^{\mathrm{T}}(-\boldsymbol{v}_i). \tag{35}$$

It is interesting that the complementary work $\tilde{w}_i^{\mathrm{C}}$ for cable member defined by (35) seems to contain the complementary strain energy w_i^{C} for truss member in the small deformation theory and the additional terms. In this section, It will be shown that (35) is derived by using the framework of complementary work in finite deformation theory.

Consider a cable member in three-dimensional space as illustrated in Fig. 3, where node (a) is fixed at $\boldsymbol{x} = \boldsymbol{0}$, and the external force $\boldsymbol{\tau}_i \in \mathbf{R}^3$ is applied at node (b). Let $\boldsymbol{h}_i^0 \in \mathbf{R}^3$ denote the nodal coordinate of node (b) at Γ^{I}, which corresponds to $\boldsymbol{\tau}_i = \boldsymbol{0}$. $\boldsymbol{r}_i \in \mathbf{R}^3$ denotes the vector of displacements of node (b).

The infinitesimal increments of work $\mathrm{d}\tilde{w}_i$ done by $\mathrm{d}\boldsymbol{r}_i$ and complementary work $\mathrm{d}\tilde{w}_i^{\mathrm{C}}$ done by $\mathrm{d}\boldsymbol{\tau}_i$, respectively, are written as

$$\mathrm{d}\tilde{w}_i = \boldsymbol{\tau}_i^{\mathrm{T}} \mathrm{d}\boldsymbol{r}_i, \tag{36}$$

$$\mathrm{d}\tilde{w}_i^{\mathrm{C}} = \boldsymbol{r}_i^{\mathrm{T}} \mathrm{d}\boldsymbol{\tau}_i. \tag{37}$$

The hysteresis independence of deformation verifies to choose a loading scenario, for example, as

$$\boldsymbol{\tau}_i(\rho) = \hat{\boldsymbol{\tau}}_i \rho \quad (0 \leqslant \rho \leqslant 1), \tag{38}$$

where $\hat{\boldsymbol{\tau}}_i \in \mathbf{R}^3$ is a constant vector, and ρ increases monotonically from $\rho = 0$ to 1. By using $\mathrm{d}\boldsymbol{\tau}_i(\rho) = \hat{\boldsymbol{\tau}}_i \mathrm{d}\rho$ and (37), we obtain

$$\tilde{w}_i^{\mathrm{C}} = \int_0^1 \boldsymbol{r}_i(\rho)^{\mathrm{T}} \hat{\boldsymbol{\tau}}_i \, \mathrm{d}\rho. \tag{39}$$

For ρ $(0 \leqslant \rho \leqslant 1)$, let $\boldsymbol{h}_i(\rho) \in \mathbf{R}^3$ and $c_i(\rho)$ denote the nodal coordinate of node (b) and the member elongation, respectively, at the equilibrium state corresponding to $\hat{\boldsymbol{\tau}}_i \rho$; i.e.,

$$\boldsymbol{h}_i(\rho) = \boldsymbol{h}_i^0 + \boldsymbol{r}_i(\rho), \quad c_i(\rho) = \|\boldsymbol{h}_i(\rho)\| - l_i^0. \tag{40}$$

Evidently, $\boldsymbol{h}_i(\rho)$ is in direction of $\boldsymbol{\tau}_i(\rho)$, from which and (38) it follows that

$$\boldsymbol{h}_i(\rho)^{\mathrm{T}} \hat{\boldsymbol{\tau}}_i = \|\boldsymbol{h}_i(\rho)\| \|\hat{\boldsymbol{\tau}}_i\| = (c_i(\rho) + l_i^0) \|\hat{\boldsymbol{\tau}}_i\|. \tag{41}$$

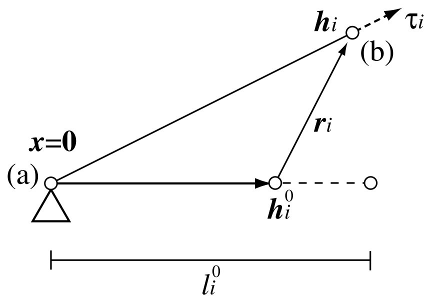

Fig. 3. Cable member.

By substituting (40) and (41) into (39), we obtain

$$\tilde{w}_i^{\mathrm{C}} = \int_0^1 (\boldsymbol{h}_i(\rho) - \boldsymbol{h}_i^0)^{\mathrm{T}} \hat{\boldsymbol{\tau}}_i \, \mathrm{d}\rho = \int_0^1 (c_i(\rho) + l_i^0) \|\hat{\boldsymbol{\tau}}_i\| \, \mathrm{d}\rho - \int_0^1 \boldsymbol{h}_i^{0^{\mathrm{T}}} \hat{\boldsymbol{\tau}}_i \, \mathrm{d}\rho = w_i^{\mathrm{C}}(\hat{\boldsymbol{\tau}}_i) + l_i^0 \|\hat{\boldsymbol{\tau}}_i\| - \boldsymbol{h}_i^{0^{\mathrm{T}}} \hat{\boldsymbol{\tau}}_i, \tag{42}$$

which is the complementary work of cable member allowing the finite deformation.

For the general formulation of complementary energy of cable network (34), we can show that (35) is induced naturally from complementary work (42). Observing that no complementary work is done during rigid-body translation of member without rotation, we can fix one of the nodes of the ith member and assign $\boldsymbol{r}_i$, $\boldsymbol{h}_i$, and $\boldsymbol{h}_i^0$ as

$$\boldsymbol{r}_i = \boldsymbol{B}_i \boldsymbol{u} - \hat{\boldsymbol{u}}_i, \quad \boldsymbol{h}_i = \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i, \quad \boldsymbol{h}_i^0 = \boldsymbol{B}_i \boldsymbol{x}^0 - \boldsymbol{b}_i^0.$$

Eq. (29) implies that, for each $i = 1, \ldots, n^{\mathrm{m}}$, the direction of $\boldsymbol{v}_i^{\mathrm{II}}$ is opposite to $\boldsymbol{h}_i^{\mathrm{II}} = \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}^{\mathrm{II}}) - \boldsymbol{b}_i$, which allows to assign as $\hat{\boldsymbol{\tau}}_i = -\boldsymbol{v}_i$. As a consequence, (42) leads to (34) along with (35). It should be emphasized that $\Pi^{\mathrm{C}}(\boldsymbol{v})$ derived by duality theory has been also derived from the concept of complementary work allowing the finite deformation.

In the case of small deformation, we may assume that $\boldsymbol{h}_i^0$ and $\hat{\boldsymbol{\tau}}_i$ have almost the same direction, and $\|\boldsymbol{h}_i^0\| \simeq l_i^0$. Accordingly, $\boldsymbol{h}_i^{0^{\mathrm{T}}} \hat{\boldsymbol{\tau}}_i \simeq l_i^0 \|\hat{\boldsymbol{\tau}}_i\|$ is obtained, from which (42) can be approximated as $\tilde{w}_i^{\mathrm{C}} \simeq w_i^{\mathrm{C}}(\hat{\boldsymbol{\tau}}_i)$, Thus, the result of (42) agrees with the well-known complementary work in the small deformation theory.

7. Illustrative example

Consider the simplest example of cable network with single-degree of freedom as shown in Fig. 4, where $n^{\mathrm{m}} = 2$, and $n^{\mathrm{d}} = 1$. Suppose that both members have the same extensional stiffness k and initial unstressed length l^0. The external force f satisfying $0 < f < 2kl^0$ is applied at node (c).

As shown in Section 4, the complementary energy function Π^{C} in (26) is different in the second term from $\Pi^{\mathrm{C}}_{\mathrm{LIN}}$ in small deformation theory. In this example, we will illustrate that the second term is necessary in finite deformation theory even without rotation of members. To this end, the reference state is given such that node (c) is located at the origin; i.e., $x^0 = 0$, $b_1 = 0$, and $b_2 = 2l^0$. The minimization problem of complementary energy will be analytically solved below, and the result will be compared with the solution to minimum total potential energy.

The problems of minimum total potential energy and complementary energy are obtained as

$$\left. \begin{array}{lll} (\Pi): \min & \Pi(u) = w_1(c_1) + w_2(c_2) - fu \\ \text{s.t.} & c_1 = |u| - l^0, \quad c_2 = |u - 2l^0| - l^0; \end{array} \right\} \tag{43}$$

$$\left. \begin{array}{lll} (\Pi^{\mathrm{C}}): \min & \Pi^{\mathrm{C}}(v_1, v_2) = \dfrac{1}{2k}(v_1^2 + v_2^2) + l^0(|v_1| + |v_2|) - 2l^0 v_2 \\ \text{s.t.} & v_1 + v_2 + f = 0, \end{array} \right\} \tag{44}$$

where $w_i(c_i)$ is defined by (8). The internal forces v_1 and v_2 are defined as shown in Fig. 5.

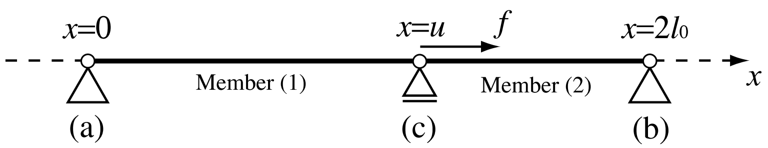

Fig. 4. The deformed state of a single-degree-of-freedom cable network.

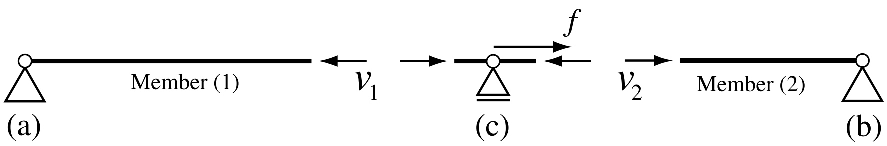

Fig. 5. Free-body scheme of node (c).

From the equilibrium equation $v_1 + v_2 + f = 0$ and the condition $f > 0$, we obtain $v_1 \leqslant 0$ or $v_2 \leqslant 0$. To solve (Π^{C}) analytically, we consider the following three cases:

(i) $v_1 \leqslant 0,\ v_2 \leqslant 0$: By using the equilibrium equation, we can eliminate v_1 from Π^{C} as

$$\Pi^{\mathrm{C}}(v_1, v_2) = \frac{1}{k}\left[v_2 - \left(kl^0 - \frac{f}{2}\right)\right]^2 + \frac{f^2}{2k} - \frac{1}{k}\left(kl^0 - \frac{f}{2}\right)^2 + l^0 f.$$

Since $kl^0 - f/2 > 0$ and $v_2 \leqslant 0$, we see that the minimum objective value is $\Pi^{\mathrm{C}}(-f, 0) = f^2/2k + l^0 f$.

(ii) $v_1 \geqslant 0,\ v_2 \leqslant 0$: Π^{C} is reduced to

$$\Pi^{\mathrm{C}}(v_1, v_2) = \frac{1}{k}\left[v_2 - \left(2kl^0 - \frac{f}{2}\right)\right]^2 + \frac{f^2}{2k} - \frac{1}{k}\left(2kl^0 - \frac{f}{2}\right)^2 - l^0 f.$$

From $v_2 = -f - v_1$ and $v_1 \geqslant 0$, we obtain $v_2 \leqslant -f$. It follow from $2kl^0 - f/2 > 0$ that Π^{C} decreases strictly on $v_2 \leqslant -f$, which implies that the minimum objective value is $\Pi^{\mathrm{C}}(0, -f) = f^2/2k + 3l^0 f$.

(iii) $v_1 \leqslant 0,\ v_2 \geqslant 0$: Π^{C} is reduced to

$$\Pi^{\mathrm{C}}(v_1, v_2) = \frac{1}{k}\left(v_2 + \frac{f}{2}\right)^2 + \frac{f^2}{4k} + l^0 f.$$

The condition $-f/2 < 0$ implies that the minimum value of Π^{C} is $\Pi^{\mathrm{C}}(-f, 0) = f^2/2k + l^0 f$.

The results of (i)–(iii) imply that the solution of (Π^{C}) is $(v_1^{\mathrm{II}}, v_2^{\mathrm{II}}) = (-f, 0)$, which is easily verified to correspond to the equilibrium state. On the other hand, the solution of (Π) is $u^{\mathrm{II}} = f/k + l^0$, which is compatible to the solution of (Π^{C}). The optimal values of Π and Π^{C} satisfy

$$\Pi(u^{\mathrm{II}}) = \frac{1}{2}k\varepsilon_1^{\mathrm{II}^2} - fu^{\mathrm{II}} = -\frac{f^2}{2k} - l^0 f = -\Pi^{\mathrm{C}}(v_1^{\mathrm{II}}, v_2^{\mathrm{II}}),$$

which illustrates the assertion of Theorem 4.3(i).

For comparison purpose, consider the classical complementary energy $\Pi^{\mathrm{C}}_{\mathrm{LIN}}$. From (33), we obtain

$$\Pi^{\mathrm{C}}_{\mathrm{LIN}}(v_1, v_2) = \frac{1}{2k}(v_1^2 + v_2^2) - 2l^0 v_2. \tag{45}$$

By using the equilibrium equation $v_1 + v_2 + f = 0$, (45) is reduced to

$$\Pi^{\mathrm{C}}_{\mathrm{LIN}}(v_2) = \frac{1}{k}\left[v_2 - \left(kl^0 - \frac{f}{2}\right)\right]^2 + \frac{f^2}{2k} - \frac{1}{k}\left(kl^0 - \frac{f}{2}\right)^2,$$

which leads to the erroneous solution $v_2 = kl^0 - f/2$. Alternatively, consider the case such that $x^0 = l^0$ and $\hat{u}_1 = \hat{u}_2 = 0$, which agrees with the assumption of small deformation. Then, (33) is reduced to

$$\Pi^{\mathrm{C}}_{\mathrm{LIN}}(v_2) = \frac{1}{k}(v_1^2 + v_2^2),$$

which leads the solution $(v_1, v_2) = (-f/2, -f/2)$; i.e., the compatible solution for a truss is obtained.

8. Remarks on the Lagrangian duality

The (extended) Lagrangian dual problem has been developed for mathematical programming problems (Mangasarian, 1969; Rockafellar, 1970) as well as variational problems (Ekeland and Temám, 1976). In this section, (Π^{C}) is revisited by using the framework of Lagrangian duality theory. In addition, we present the unified view point for several complementary energy principles ever addressed in the literature for finite dimensional structures. It is interesting that the variety of approaches is understood as the different formulations of Lagrangian. It is remarkable that our approach does not have any difficulty or ambiguity of inversion of constitutive law.

We start with (Π); i.e., the Lagrangian of (Π) can be defined as

$$L^{\Pi}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\lambda}) = \sum_{i=1}^{n^{\mathrm{m}}} w_i(c_i) - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} - \sum_{i=1}^{n^{\mathrm{m}}} \lambda_i \left[c_i + l_i^0 - \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\|\right],$$

where λ_i $(i = 1,\dots,n^{\mathrm{m}})$ are the Lagrangian multipliers, and $\boldsymbol{\lambda} = (\lambda_i) \in \mathbf{R}^{n^{\mathrm{m}}}$. Note that L^{Π} corresponds to the Hu–Washizu functional, which is well-known in the continuum mechanics. We see that (Π) can be alternatively written as

$$\mathscr{P}(L^{\Pi}) : \min_{\boldsymbol{c},\boldsymbol{u}} \quad \sup\{L^{\Pi}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\lambda}) | \boldsymbol{\lambda} \in \mathbf{R}^{n^{\mathrm{m}}}\}, \tag{46}$$

where

$$\sup\{L^{\Pi}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\lambda}) | \boldsymbol{\lambda} \in \mathbf{R}^{n^{\mathrm{m}}}\} = \begin{cases} \sum_{i=1}^{n^{\mathrm{m}}} w_i(c_i) - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} & (c_i + l_i^0 - \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\|,\ i = 1,\dots,n^{\mathrm{m}}), \\ +\infty & (\text{otherwise}). \end{cases}$$

The Lagrangian dual problem is obtained by replacing min-sup with max-inf in (46); i.e.,

$$\mathscr{D}(L^{\Pi}) : \max_{\boldsymbol{\lambda}} \quad \inf\{L^{\Pi}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\lambda}) | \boldsymbol{c} \in \mathbf{R}^{n^{\mathrm{m}}},\ \boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}\}. \tag{47}$$

Unfortunately, L^{Π} is a nonconvex and nonsmooth function of $\boldsymbol{u}$. Accordingly, it is difficult to calculate the infimum in (47) explicitly, and there exists the duality gap between $\mathscr{P}(L^{\Pi})$ and $\mathscr{D}(L^{\Pi})$ generally. This explains, from the view point of duality theory, that the standard approach using $\mathscr{D}(L^{\Pi})$ fails to derive the truly complementary energy principle.

Consider (P) instead of (Π). By using the self-duality of second-order cone introduced in (1), the Lagrangian of (P) can be defined as

$$L^{P}(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) = \begin{cases} \displaystyle\sum_{i=1}^{n^{\mathrm{m}}} \frac{1}{2} k_i y_i^2 - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} - \sum_{i=1}^{n^{\mathrm{m}}} q_i (y_i + l_i^0) & \\ \quad - \sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{v}_i^{\mathrm{T}} [\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i] & (q_i \geqslant \|\boldsymbol{v}_i\|,\ i = 1,\dots,n^{\mathrm{m}}), \\ +\infty & (\text{otherwise}), \end{cases}$$

where $\boldsymbol{q} = (q_i) \in \mathbf{R}^{n^{\mathrm{m}}}$ and $\boldsymbol{v} = (\boldsymbol{v}_1,\dots,\boldsymbol{v}_{n^{\mathrm{m}}}) \in \mathbf{R}^{3n^{\mathrm{m}}}$ are the Lagrangian multipliers. Indeed, (P) is equivalent to

$$\mathscr{P}(L^{P}) : \min_{\boldsymbol{y},\boldsymbol{u}} \quad \sup\{L^{P}(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) | \boldsymbol{q} \in \mathbf{R}^{n^{\mathrm{m}}},\ \boldsymbol{v} \in \mathbf{R}^{3n^{\mathrm{m}}}\}, \tag{48}$$

which validates that L^{P} can be regarded as the extended Lagrangian of (P) (Ekeland and Temám, 1976). It is remarkable to note that L^{P} is a linear function of $\boldsymbol{u}$, even if (P) is a nonlinear programming problem. The Lagrangian dual problem is now defined as

$$\mathscr{D}(L^{P}) : \max_{\boldsymbol{q},\boldsymbol{v}} \quad \inf\{L^{P}(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) | \boldsymbol{y} \in \mathbf{R}^{n^{\mathrm{m}}},\ \boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}\}. \tag{49}$$

Note that L^P is a convex and smooth function with respect to $\boldsymbol{y}$ and $\boldsymbol{u}$, from which the problem (49) is reduced to

$$\left.\begin{array}{lll}\mathscr{D}(L^P): \max & L^P(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) \\ \text{s.t.} & \nabla_{\boldsymbol{y}}L^P = \mathbf{0}, \quad \nabla_{\boldsymbol{u}}L^P = \mathbf{0}, \quad q_i \geqslant \|\boldsymbol{v}_i\| \quad (i=1,\ldots,n^{\mathrm{m}}).\end{array}\right\} \tag{50}$$

Here, $\nabla_{\boldsymbol{y}}L^P = \mathbf{0}$ is reduced to

$$q_i = k_i y_i \quad (i=1,\ldots,n^{\mathrm{m}}). \tag{51}$$

Eq. (51) is the constitutive law in terms of member elongation and axial force, which is analogous to that in terms of Green–Lagrange strain tensor and the second Piola–Kirchhoff stress tensor used in the continuum mechanics. It is obvious that (51) has unique inversion. By using this property, we can eliminate y_i from the objective function of $\mathscr{D}(L^P)$. Similarly, $\boldsymbol{u}$ can be eliminated by using $\nabla_{\boldsymbol{u}}L^P = \mathbf{0}$, because L^P is a linear function with respect to $\boldsymbol{u}$. Thus, $\mathscr{D}(L^P)$ can be reformulated into the form without $\boldsymbol{y}$ and $\boldsymbol{u}$, which is the expected property for the truly complementary energy principle. Actually, it is easy to see that $\mathscr{D}(L^P)$ coincides with (D) in Section 9.2, which is equivalent to (Π^{C}) (see Lemma 9.2).

In Section 4, we derived (D_{S}) by using the well-established results about the duality of SOCP. The series of lemmas and theorem in Section 4 may also be obtained by using the results of (extended) Lagrangian duality theory (Ekeland and Temám, 1976, Chapter VI). The nice separable duality property of SOCP can be understood from the characteristics of the Lagrangian such that L^P is a linear function of $\boldsymbol{u}$ with the nonlinear subsidiary conditions of the Lagrangian multipliers. By utilizing this type of Lagrangian, the authors derived the necessary and sufficient conditions for optimality of the structural optimization problem of trusses under the fundamental frequency constraints (Kanno and Ohsaki, 2001).

Letting $\boldsymbol{r}_i = \boldsymbol{B}_i\boldsymbol{u} - \hat{\boldsymbol{u}}_i \in \mathbf{R}^3$ and $\boldsymbol{h}_i^0 = \boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{b}_i^0 \in \mathbf{R}^3$ $(i=1,\ldots,n^{\mathrm{m}})$, we see $c_i = \|\boldsymbol{r}_i + \boldsymbol{h}_i^0\| - l_i^0$. Define a function w_i^r as

$$w_i^r(\boldsymbol{r}_i) = w_i(\|\boldsymbol{r}_i + \boldsymbol{h}_i^0\| - l_i^0). \tag{52}$$

Then, (Π) is alternatively written as

$$\left.\begin{array}{lll}\mathscr{P}(L^r): \min & \sum_{i=1}^{n^{\mathrm{m}}} w_i^r(\boldsymbol{r}_i) - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} \\ \text{s.t.} & \boldsymbol{r}_i = \boldsymbol{B}_i\boldsymbol{u} - \hat{\boldsymbol{u}}_i \quad (i=1,\ldots,n^{\mathrm{m}}),\end{array}\right\} \tag{53}$$

along with the Lagrangian

$$L^r(\boldsymbol{r},\boldsymbol{u},\boldsymbol{\tau}) = \sum_{i=1}^{n^{\mathrm{m}}} w_i^r(\boldsymbol{r}_i) - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} - \sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{\tau}_i^{\mathrm{T}}(\boldsymbol{r}_i - \boldsymbol{B}_i\boldsymbol{u} + \hat{\boldsymbol{u}}_i), \tag{54}$$

where $\boldsymbol{\tau}_i \in \mathbf{R}^3$ $(i=1,\ldots,n^{\mathrm{m}})$ are the Lagrangian multipliers, $\boldsymbol{\tau} = (\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_{n^{\mathrm{m}}}) \in \mathbf{R}^{3n^{\mathrm{m}}}$, and $\boldsymbol{r} = (\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{n^{\mathrm{m}}}) \in \mathbf{R}^{3n^{\mathrm{m}}}$. Mikkola (1989) investigated L^r for truss structures. For continuum, this type of Lagrangian can be found in various literature (see, e.g., Koiter, 1976, (6.4)). In connection with the Lagrangian dual problem, we see that L^r attains $\inf\{L^r | \boldsymbol{r} \in \mathbf{R}^{3n^{\mathrm{m}}},\ \boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}\}$ only if

$$\nabla_{\boldsymbol{u}}L^r = \mathbf{0}, \tag{55}$$

$$\nabla_{\boldsymbol{r}_i}L^r = \mathbf{0} \quad (i=1,\ldots,n^{\mathrm{m}}) \tag{56}$$

are satisfied. Eq. (56) is reduced to

$$\boldsymbol{\tau}_i = \frac{\partial w_i^r(\boldsymbol{r}_i)}{\partial \boldsymbol{r}_i} \quad (i=1,\ldots,n^{\mathrm{m}}). \tag{57}$$

For the ith member, (57) is the constitutive law in terms of the nodal displacement vector and internal force vector, which is analogous to that relating the displacement gradient and the first Piola–Kirchhoff stress tensor in the case of continuum. Assuming that (57) has an inversion, we can define the function $\widehat{w}_i^{\mathrm{C}}(\boldsymbol{\tau}_i)$ only in terms of $\boldsymbol{\tau}_i$ as

$$\widehat{w}_i^{\mathrm{C}}(\boldsymbol{\tau}_i) = \boldsymbol{\tau}_i^{\mathrm{T}}\boldsymbol{r}_i - w_i^r(\boldsymbol{r}_i), \tag{58}$$

which is the Legendre transformation of w_i^r. By substituting (55) and (58) into (54), we can formulate the dual problem of $\mathscr{P}(L^r)$ as

$$\left.\begin{array}{lll} \widehat{\mathscr{D}}(L^r): \max & -\sum_{i=1}^{n^{\mathrm{m}}}\hat{w}_i^{\mathrm{C}}(\boldsymbol{\tau}_i) - \sum_{i=1}^{n^{\mathrm{m}}}\hat{\boldsymbol{u}}_i^{\mathrm{T}}\boldsymbol{\tau}_i \\ \text{s.t.} & \sum_{i=1}^{n^{\mathrm{m}}}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{\tau}_i - \boldsymbol{f} = \boldsymbol{0}, \end{array}\right\} \tag{59}$$

which corresponds to the classical Lagrangian dual problem (Mangasarian, 1969). Therefore, the strong duality does not hold between $\mathscr{P}(L^r)$ and $\widehat{\mathscr{D}}(L^r)$ generally. Since L^r is a linear function of $\boldsymbol{u}$, the constraints of $\widehat{\mathscr{D}}(L^r)$ contain only stress components $\boldsymbol{\tau}_i$ as variables. However, as clarified by Koiter (1976), (57) does not have a unique inversion in general. Hence, the objective function of $\widehat{\mathscr{D}}(L^r)$ is a multi-valued function (see Mikkola (1989) for the case of trusses). Even in the case of cable network, (57) has the multi-valued inversion at $\boldsymbol{\tau}_i = \boldsymbol{0}$. This is the difficulty of approach based on the Legendre transformation with the standard type of Lagrangian.

Recently, by using the (Fenchel's) conjugate transformation, Atai and Steigmann (1997) proposed the minimum principle of complementary energy for cable networks. For the purpose of comparison, we show that the same result can be obtained based on the Lagrangian duality approach. In Atai and Steigmann (1997), a cable network is modeled as an assemblage of one-dimensional continuum, but they restricted themselves to the case where the external loads are applied only at nodes. Accordingly, we can discuss the finite dimensional model of cable networks without loss of generality.

In order to formulate $\widehat{\mathscr{D}}(L^r)$, we used the necessary conditions (55) and (56) for infimum of L^r. On the contrary, by calculating the infimum directly, we obtain

$$\begin{aligned} &\inf\{L^r(\boldsymbol{r},\boldsymbol{u},\boldsymbol{\tau}) \mid \boldsymbol{r} \in \mathbf{R}^{3n^{\mathrm{m}}},\ \boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}\} \\ &= \sum_{i=1}^{n^{\mathrm{m}}} \inf\{\boldsymbol{\tau}_i^{\mathrm{T}}\boldsymbol{r}_i - w_i^r(\boldsymbol{r}_i) \mid \boldsymbol{r}_i \in \mathbf{R}^3\} + \inf\left\{\left(\sum_{i=1}^{n^{\mathrm{m}}}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{\tau}_i - \boldsymbol{f}\right)^{\mathrm{T}}\boldsymbol{u} \,\middle|\, \boldsymbol{u} \in \mathbf{R}^{n^{\mathrm{d}}}\right\} - \sum_{i=1}^{n^{\mathrm{m}}}\hat{\boldsymbol{u}}_i^{\mathrm{T}}\boldsymbol{\tau}_i \\ &= \begin{cases} -\sum_{i=1}^{n^{\mathrm{m}}} w_i^*(\boldsymbol{\tau}_i) - \sum_{i=1}^{n^{\mathrm{m}}}\hat{\boldsymbol{u}}_i^{\mathrm{T}}\boldsymbol{\tau}_i & \left(\sum_{i=1}^{n^{\mathrm{m}}}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{\tau}_i - \boldsymbol{f} = \boldsymbol{0}\right), \\ -\infty & (\text{otherwise}), \end{cases} \end{aligned} \tag{60}$$

where w_i^* is the (Fenchel's) conjugate transformation of w_i^r defined by (Rockafellar, 1970)

$$w_i^*(\boldsymbol{\tau}_i) = \sup\{\boldsymbol{\tau}_i^{\mathrm{T}}\boldsymbol{r}_i - w_i^r(\boldsymbol{r}_i) \mid \boldsymbol{r}_i \in \mathbf{R}^3\}. \tag{61}$$

By using (60) and (61), the (extended) Lagrangian dual problem of $\mathscr{P}(L^r)$ is obtained as

$$\left.\begin{array}{lll} \mathscr{D}(L^r): \max & -\sum_{i=1}^{n^{\mathrm{m}}} w_i^*(\boldsymbol{\tau}_i) - \sum_{i=1}^{n^{\mathrm{m}}}\hat{\boldsymbol{u}}_i^{\mathrm{T}}\boldsymbol{\tau}_i \\ \text{s.t.} & \sum_{i=1}^{n^{\mathrm{m}}}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{\tau}_i - \boldsymbol{f} = \boldsymbol{0}. \end{array}\right\} \tag{62}$$

This formulation is similar to that of Atai and Steigmann (1997), but they used the deformation gradient $\boldsymbol{h}_i = \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i \in \mathbf{R}^3$ instead of displacement gradient $\boldsymbol{r}_i$. Note that the strong duality holds between

$\mathscr{P}(L^r)$ and $\mathscr{D}(L^r)$ if w_i^r is convex (Rockafellar, 1970). Otherwise, there exists positive duality gap generally, and an optimal solution of $\mathscr{D}(L^r)$ does not satisfy the equilibrium conditions.

Note that w_i^* is always a (single-valued) function of $\boldsymbol{\tau}_i$. However, this does not directly imply that w_i^* can be written explicitly in a simple algebraic form only in terms of $\boldsymbol{\tau}_i$. Unfortunately, Atai and Steigmann (1997) did not present the explicit formulation of complementary energy.

By comparing L^{Π}, L^P, and L^r, we see that our approach presented in this paper is independent of the concepts ever addressed. Namely, the ambiguity in inversion of constitutive law is successfully avoided. Moreover, the complementary energy has been obtained explicitly, which is the practical advantage of our approach.

9. Proofs of lemmas

9.1. Proof of Lemma 3.3

Lemma 3.3 is shown by converting (P) to (P_{S}).

Proof. Introducing the auxiliary variables t_i $(i = 1, \dots, n^{\mathrm{m}})$, we can reformulate (P) as

$$\left.\begin{array}{ll} \min & \phi_{\mathrm{S}}^P(\boldsymbol{u}, \boldsymbol{t}) = \sum_{i=1}^{n^{\mathrm{m}}} t_i - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u} \\ \text{s.t.} & t_i \geqslant \dfrac{k_i}{2} y_i^2, \quad y_i + l_i^0 \geqslant \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}). \end{array}\right\} \tag{63}$$

We easily see that

$$t_i \geqslant \frac{k_i}{2} y_i^2 \iff \frac{t_i}{2k_i} + 1 \geqslant \left\| \begin{pmatrix} \frac{t_i}{2k_i} - 1 \\ y_i \end{pmatrix} \right\|, \tag{64}$$

which implies that the problem (63) is reduced to (P_{S}). An optimal solution of the problem (63) satisfies (64) in equality for each $i = 1, \dots, n^{\mathrm{m}}$, from which Lemma 3.3(i) follows. Lemma 3.3(ii) is immediately obtained from the assertion (i). □

9.2. Proof of 4.1

Consider the following problem:

$$\left.\begin{array}{ll} (D): \max & \phi^D(\boldsymbol{q}, \boldsymbol{v}) = -\sum_{i=1}^{n^{\mathrm{m}}} \frac{q_i^2}{2k_i} - \sum_{i=1}^{n^{\mathrm{m}}} l_i^0 q_i - \sum_{i=1}^{n^{\mathrm{m}}} (\boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{b}_i)^{\mathrm{T}} \boldsymbol{v}_i \\ \text{s.t.} & \sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i^{\mathrm{T}} \boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0}, \quad q_i \geqslant \|\boldsymbol{v}_i\| \quad (i = 1, \dots, n^{\mathrm{m}}), \end{array}\right\} \tag{65}$$

where independent variables are $\boldsymbol{q} = (q_i) \in \mathbf{R}^{n^{\mathrm{m}}}$ and $\boldsymbol{v} = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n^{\mathrm{m}}}) \in \mathbf{R}^{3n^{\mathrm{m}}}$. We show Lemma 4.1 by two steps; i.e., we investigate the relationship between (D_{S}) and (D), and between (D) and (Π^{C}). The following lemma provides us with the first step:

Lemma 9.1 (Relation between (D_{S}) and (D)). *$(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$ is an optimal solution of (D) if and only if $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{\xi}})$ satisfying*

$$\bar{\xi}_{i0} = \frac{\bar{q}_i^2}{4k_i} + k_i, \quad \bar{\xi}_{i1} = -\frac{\bar{q}_i^2}{4k_i} + k_i, \quad \bar{\xi}_{i2} = -\bar{q}_i \quad (i = 1, \dots, n^{\mathrm{m}}), \tag{66}$$

is an optimal solution of (D_{S}). Moreover, the optimal objective values of (D) and (D_{S}) coincide.

Proof. By using the constraint conditions of $(D_{\rm S})$

$$\xi_{i0} + \xi_{i1} = 2k_i, \tag{67}$$

$$\xi_{i0} \geqslant \|(\xi_{i1}, \xi_{i2})^{\rm T}\|, \tag{68}$$

$$q_i + \xi_{i2} = 0, \tag{69}$$

and introducing new variables $t_i^{\rm C}$, we can eliminate $(\xi_{i0}, \xi_{i1}, \xi_{i2})$ from $(D_{\rm S})$. (67) verifies that ξ_{i0} and ξ_{i1} are rewritten as

$$\xi_{i0} = \frac{t_i^{\rm C}}{2} + k_i, \quad \xi_{i1} = -\frac{t_i^{\rm C}}{2} + k_i. \tag{70}$$

By using (69) and (70), (68) is reduced to

$$\frac{t_i^{\rm C}}{2} + k_i \geqslant \left\| \begin{pmatrix} -\frac{t_i^{\rm C}}{2} + k_i \\ -q_i \end{pmatrix} \right\| \Longleftrightarrow t_i^{\rm C} \geqslant \frac{q_i^2}{2k_i}, \tag{71}$$

and the first term of $\phi_{\rm S}^D$ in $(D_{\rm S})$ is reduced to

$$\sum_{i=1}^{n^{\rm m}} (-\xi_{i0} + \xi_{i1}) = -\sum_{i=1}^{n^{\rm m}} t_i^{\rm C}. \tag{72}$$

From (71) and (72), $(D_{\rm S})$ is reformulated as

$$\left.\begin{array}{ll} \max & -\sum_{i=1}^{n^{\rm m}} t_i^{\rm C} - \sum_{i=1}^{n^{\rm m}} l_i^0 q_i - \sum_{i=1}^{n^{\rm m}} (\boldsymbol{B}_i \boldsymbol{x}^0 - \boldsymbol{b}_i)^{\rm T} \boldsymbol{v}_i \\ \text{s.t.} & \sum_{i=1}^{n^{\rm m}} \boldsymbol{B}_i^{\rm T} \boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0}, \\ & t_i^{\rm C} \geqslant \dfrac{q_i^2}{2k_i}, \quad q_i \geqslant \|\boldsymbol{v}_i\| \quad (i = 1, \ldots, n^{\rm m}). \end{array}\right\} \tag{73}$$

Observing that the objective function is a strictly decreasing function of $t_i^{\rm C}$, any optimal solutions of the problem (73) satisfy

$$t_i^{\rm C} = q_i^2 / 2k_i \quad (i = 1, \ldots, n^{\rm m}). \tag{74}$$

Therefore, the problem (73) is equivalent to (D). Conditions (66) in Lemma 9.1 can be obtained from (69), (70) and (74). □

The following lemma relates optimal solutions $\boldsymbol{v}$ of $(\Pi^{\rm C})$ and $(\boldsymbol{q}, \boldsymbol{v})$ of (D):

Lemma 9.2 (Relation between (D) and $(\Pi^{\rm C})$). *$\bar{\boldsymbol{v}}$ is an optimal solution of $(\Pi^{\rm C})$ if and only if $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$ satisfying*

$$\bar{q}_i = \|\bar{\boldsymbol{v}}_i\| \quad (i = 1, \ldots, n^{\rm m}), \tag{75}$$

is an optimal solution of (D), where $\Pi^{\rm C}(\bar{\boldsymbol{v}}) = -\phi^D(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$.

Proof. For an optimal solution $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$ of (D), (75) is satisfied because $-l_i^0 q_i$ decreases strictly in the feasible region of (D). Accordingly, (D) is reduced to $(\Pi^{\rm C})$ by converting the constraints $q_i \geqslant \|\boldsymbol{v}_i\|$ $(i = 1, \ldots, n^{\rm m})$ to the equalities $q_i = \|\boldsymbol{v}_i\|$, and by changing the sign of the objective function to transform maximization to minimization, which completes the proof. □

Proof of Lemma 4.1. Lemma 4.1 follows from Lemmas 9.1 and 9.2 immediately. □

10. Proof of Theorem 4.3

The discussion in this section is based on the framework of SOCP duality (Theorem 2.2). The following two lemmas should be prepared to prove Theorem 4.3:

Lemma 10.1 (Strong duality of $(P_{\rm S})$ and $(D_{\rm S})$). *Under Assumption* 3.1,

(i) $(P_{\rm S})$ *and* $(D_{\rm S})$ *have optimal solutions* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{t}})$ *and* $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{\xi}})$, *respectively, and* $\phi_{\rm S}^{P}(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{t}}) = \phi_{\rm S}^{D}(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{\xi}})$.

(ii) *feasible solution* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{t}})$ *and* $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{\xi}})$ *of* $(P_{\rm S})$ *and* $(D_{\rm S})$, *respectively, are optimal solutions if and only if they satisfy*

$$\bar{q}_i(\bar{y}_i + l_i^0) + \bar{\boldsymbol{v}}_i^{\rm T}[\boldsymbol{B}_i(\boldsymbol{x}^0 + \bar{\boldsymbol{u}}) - \boldsymbol{b}_i] = 0 \quad (i = 1, \dots, n^{\rm m}), \tag{76}$$

$$\bar{\xi}_{i0}\left(\frac{\bar{t}_i}{2k_i} + 1\right) + \bar{\xi}_{i1}\left(\frac{\bar{t}_i}{2k_i} - 1\right) + \bar{\xi}_{i2}\bar{y}_i = 0 \quad (i = 1, \dots, n^{\rm m}). \tag{77}$$

Proof. Assumption 3.1 guarantees Lemma 4.2(i), from which it is not difficult to see that the matrices $\boldsymbol{A}_i$ $(i = 1, \dots, 2n^{\rm m})$ defined by (17) satisfy Assumption 2.1(i).

Let $\mathscr{F}^0(P_{\rm S}) \subseteq \mathbf{R}^{2n^{\rm m}+n^{\rm d}}$ and $\mathscr{F}^0(D_{\rm S}) \subseteq \mathbf{R}^{7n^{\rm m}}$ denote the sets of interior-feasible solutions of $(P_{\rm S})$ and $(D_{\rm S})$, respectively. By using Lemma 4.2(ii), we shall show that Assumption 3.1 guarantees Assumption 2.1(ii) is satisfied. By using (64), we obtain

$$\mathscr{F}^0(P_{\rm S}) = \left\{ (\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{t}) \middle| t_i > \frac{1}{2}k_i y_i^2,\ y_i + l_i^0 > \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \boldsymbol{b}_i\| \ (i = 1, \dots, n^{\rm m}) \right\},$$

where both t_i and y_i are not bounded from above. Therefore, $\mathscr{F}^0(P_{\rm S}) \neq \emptyset$ is always satisfied. From (69)–(71), we obtain

$$\mathscr{F}^0(D_{\rm S}) = \left\{ (\boldsymbol{q}, \boldsymbol{v}, \xi) \middle| \sum_{i=1}^{n^{\rm m}} \boldsymbol{B}_i^{\rm T}\boldsymbol{v}_i + \boldsymbol{f} = \boldsymbol{0},\ (69),\ (70),\ t_i^{\rm C} > \frac{q_i^2}{2k_i},\ q_i > \|\boldsymbol{v}_i\| \ (i = 1, \dots, n^{\rm m}) \right\},$$

where both $t_i^{\rm C}$ and q_i are not bounded from above. Accordingly, $\mathscr{F}^0(\Pi^{\rm C}) \neq \emptyset$ implies $\mathscr{F}^0(D_{\rm S}) \neq \emptyset$, from which and Lemma 4.2(ii) it follows that Assumption 3.1 guarantees $\mathscr{F}^0(D_{\rm S}) \neq \emptyset$.

Consequently, the assumption in Lemma 10.1 is equivalent to Assumption 2.1. Lemma 10.1(i) follows from Theorem 2.2(i) immediately. By substituting (16) and (19) into (5), we obtain (76) and (77). Therefore, Lemma 10.1(ii) follows from Theorem 2.2(ii)b. □

As a consequence of Lemma 10.1, the following result about the duality between (P) and (D) is obtained:

Lemma 10.2 (Strong duality of (P) and (D)). *Under Assumption* 3.1,

(i) (P) *and* (D) *have optimal solutions* $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$ *and* $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$, *respectively, and*

$$\phi^P(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}) = \phi^D(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}). \tag{78}$$

(ii) $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$ *and* $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$ *are optimal solutions of* (P) *and* (D), *respectively, if and only if they satisfy*

$$\bar{q}_i = k_i\bar{y}_i \quad (i = 1, \dots, n^{\rm m}), \tag{79}$$

$$\sum_{i=1}^{n^{\mathrm{m}}} \boldsymbol{B}_i^{\mathrm{T}} \bar{\boldsymbol{v}}_i + \boldsymbol{f} = 0, \tag{80}$$

$$\bar{y}_i + l_i^0 \geqslant \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \bar{\boldsymbol{u}}) - \boldsymbol{b}_i\| \quad (i = 1, \ldots, n^{\mathrm{m}}), \tag{81}$$

$$\bar{q}_i \geqslant \|\bar{\boldsymbol{v}}_i\| \quad (i = 1, \ldots, n^{\mathrm{m}}), \tag{82}$$

$$\bar{q}_i(\bar{y}_i + l_i^0) + \bar{\boldsymbol{v}}_i^{\mathrm{T}}[\boldsymbol{B}_i(\boldsymbol{x}^0 + \bar{\boldsymbol{u}}) - \boldsymbol{b}_i] = 0 \ (i = 1, \ldots, n^{\mathrm{m}}). \tag{83}$$

Proof

(i) Recall that we have investigated the relation between (P) and (P_{S}) in Lemma 3.3, and the relation between (D_{S}) and (D) was given in Lemma 9.1. As a consequence of these lemmas, Lemma 10.1(i) follows from Lemma 10.1(i).

(ii) It follows from Lemmas 3.3, 9.1, and 10.1 that $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})$ and $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}})$ are optimal solutions of (P) and (D), respectively, if and only if feasible solutions $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}}, \bar{\boldsymbol{t}})$ and $(\bar{\boldsymbol{q}}, \bar{\boldsymbol{v}}, \bar{\boldsymbol{\xi}})$ of (P_{S}) and (D_{S}) satisfy (13), (66), (76) and (77). Therefore, we only have to show that the conditions (13), (66) and (77) are equivalent to (79). By substituting (13) and (66) into the left-hand side of (77), we obtain

$$\begin{aligned} \bar{\xi}_{i0}\left(\frac{\bar{t}_i}{2k_i} + 1\right) + \bar{\xi}_{i1}\left(\frac{\bar{t}_i}{2k_i} - 1\right) + \bar{\xi}_{i2}\bar{y}_i &= \left(\frac{\bar{q}_i^2}{4k_i} + k_i\right)\left(\frac{\bar{y}_i^2}{4} + 1\right) + \left(-\frac{\bar{q}_i^2}{4k_i} + k_i\right)\left(\frac{\bar{y}_i^2}{4} - 1\right) + (-\bar{q}_i)\bar{y}_i \\ &= \frac{\bar{q}_i^2}{2k_i} + \frac{k_i\bar{y}_i^2}{2} - \bar{q}_i\bar{y}_i = \frac{1}{2k_i}(k_i\bar{y}_i - \bar{q}_i)^2, \end{aligned}$$

which completes the proof. □

Note that (79)–(83) can be also obtained by using the KKT conditions for convex nonsmooth optimization problems (Rockafellar, 1970), where (83) corresponds to the complementarity condition.

Proof of Theorem 4.3. It follows from Lemma 3.2, Lemma 9.2 and Lemma 10.2(ii) that $(\boldsymbol{c}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$ and $\boldsymbol{v}^{\mathrm{II}}$ are optimal solutions of (Π) and (Π^{C}), respectively, if and only if $(\boldsymbol{c}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$, $\bar{\boldsymbol{y}}$, and $(\bar{\boldsymbol{q}}, \boldsymbol{v}^{\mathrm{II}})$ defined by (11) and (75) satisfy (6) and (79)–(83). After simple manipulation, the latter condition is reduced to

$$(11), \quad (27), \quad (28), \quad (30), \tag{84}$$

$$\bar{q}_i = k_i\bar{y}_i = \|\boldsymbol{v}_i^{\mathrm{II}}\| \quad (i = 1, \ldots, n^{\mathrm{m}}), \tag{85}$$

$$\bar{q}_i(\bar{y}_i + l_i^0) + \boldsymbol{v}_i^{\mathrm{II}^{\mathrm{T}}}\boldsymbol{h}_i^{\mathrm{II}} = 0 \quad (i = 1, \ldots, n^{\mathrm{m}}). \tag{86}$$

It suffices to show that (29) is equivalent to (11), (85) and (86).

For $-l_i^0 \leqslant c_i^{\mathrm{II}} < 0$, (11) implies $\bar{y}_i = 0$, from which and (85) we obtain $\boldsymbol{v}_i^{\mathrm{II}} = \boldsymbol{0}$; i.e., (29) is satisfied. Alternatively, for $c_i^{\mathrm{II}} \geqslant 0$, (11) implies $\bar{y}_i = c_i^{\mathrm{II}}$. It follows from (27) and (28) that $\bar{y}_i + l_i^0 = \|\boldsymbol{h}_i^{\mathrm{II}}\|$. By substituting this identity into (85) and using (86), we obtain $\|\boldsymbol{v}_i^{\mathrm{II}}\|\|\boldsymbol{h}_i^{\mathrm{II}}\| + \boldsymbol{v}_i^{\mathrm{II}^{\mathrm{T}}}\boldsymbol{h}_i^{\mathrm{II}} = 0$, which leads to

$$\boldsymbol{v}_i^{\mathrm{II}} = -\|\boldsymbol{v}_i^{\mathrm{II}}\|\frac{\boldsymbol{h}_i^{\mathrm{II}}}{\|\boldsymbol{h}_i^{\mathrm{II}}\|}, \tag{87}$$

where $\boldsymbol{h}_i^{\mathrm{II}} \neq 0$ because of $c_i^{\mathrm{II}} \geqslant 0$. It follows from (7) and (85) that $\|\boldsymbol{v}_i^{\mathrm{II}}\| = k_i c_i^{\mathrm{II}} = \sigma_i(c_i^{\mathrm{II}})$. Consequently, the triple $(c_i^{\mathrm{II}}, \boldsymbol{v}_i^{\mathrm{II}}, \boldsymbol{h}_i^{\mathrm{II}})$ satisfies (29) if (11), (85) and (86) hold. Conversely, it is easy to see that (11), (85) and (86) are satisfied if (29) holds. □

11. Conclusions

The minimum principle of complementary energy has been established for cable networks in geometrically nonlinear elasticity.

The minimization problem (Π) of total potential energy for cable networks has been first formulated allowing finite deformation, and its SOCP formulation (P_S) has been presented. Based on the duality of a pair of primal–dual SOCP problems, the minimum principle of complementary energy is established; i.e., the minimization problem of complementary energy (Π^C) is simply derived from the dual SOCP problem (D_S) of (P_S). From the strong duality of SOCP, we have shown the strong duality theorem between (Π) and (Π^C), which guarantees that an optimal solution of (Π^C) corresponds to a set of internal force vectors at the equilibrium state.

It is known that, in general, the complementary energy function expressed only by stress components cannot be uniquely determined, and minimum principle cannot be established even if the equilibrium state is stable. On the contrary, it has been shown in this paper that cable networks have the unique complementary energy function and the minimum principle of complementary energy can be established irrespective of the stability of equilibrium state. Therefore, the presented principle may be actually useful in practical application such as the force method. Moreover, the obtained complementary energy has been interpreted physically based on the concept of complementary work in finite deformation.

The existence and uniqueness of the solution to the minimum complementary energy have been investigated. Based on the graph theory, we have presented the sufficient condition for existence of the solution to the problem of minimum complementary energy allowing large deformation. This condition is related to the topology and the support condition of cable networks, but is independent of the geometry of structures and the property of external loads. From the strict convexity of the complementary energy function, the sufficient condition for the uniqueness of equilibrium configuration of a cable network has also been presented, which is also independent of its geometry and magnitudes of axial forces.

Existing approaches to complementary energy principle have been compared from the unified view point of Lagrangian duality. It has been also shown that (D) can be obtained based on the framework of Lagrangian duality by utilizing the self-dual property of second-order cone, which has clarified how our approach can avoid the difficulties of coupling of stress and displacement and uniqueness of inversion of constitutive law.

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